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A Mal'cev characterization of tolerance regularity

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A variety $\mathcal{V}$ of algebras is *regular* if it contains only regular algebras, i.e. if any two congruences on $\mathfrak{A}\in\mathcal{V}$ coincide whenever they have a congruence class in common. The regularity of varieties is a Mal'cev condition, see [7], [11], [12]. A *tolerance* T on an algebra $\mathfrak{A}=(A, F)$ is a reflexive and symmetric binary relation on A satisfying the Substitution Property with respect to all operations of $\mathfrak{A}$; this means that

$$\langle f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)\rangle \in T$$

for each n-ary $f\in F$ whenever $\langle a_i, b_i\rangle\in T$ for $a_i, b_i\in A$ $(i=1, \ldots, n)$. This notion comes from that of congruence by omitting the requirement of transitivity. The set $\mathrm{LT}(\mathfrak{A})$ of all tolerances on an algebra $\mathfrak{A}$ forms an algebraic lattice with respect to set inclusion (see [2], [6]). Hence, we can introduce the following concepts. If $a, b\in A$ and $M\subseteq A\times A$, denote by $T(a, b)$ or $T(M)$ the least tolerance on $\mathfrak{A}$ containing the pair $\langle a, b\rangle$ or the set M, respectively.

Let $T\in\mathrm{LT}(\mathfrak{A})$. Call $[a]_T=\{b\in A; \langle a, b\rangle\in T\}$ the *tolerance class of T containing $a\in A$*. This generalizes the concept of a congruence class; other generalizations can be found in [3], [4], [5].

Definition 1. An algebra $\mathfrak{A}$ is *tolerance regular* if any two tolerances on $\mathfrak{A}$ coincide whenever they have a tolerance class in common. A variety $\mathcal{V}$ of algebras is *tolerance regular* if each $\mathfrak{A}\in\mathcal{V}$ has this property.

Let $T\in\mathrm{LT}(\mathfrak{A})$ and let $[a]_T$ be a tolerance class of T. Denote by $\mathrm{Tol}\,\{[a]_T\}$ the least tolerance on $\mathfrak{A}$ having a tolerance class equal to $[a]_T$. Clearly $\mathrm{Tol}\,\{[a]_T\}=$ $=T(M)$ for $M=\{a\}\times[a]_T$.

Lemma 1. *Let $\mathfrak{A}=(A, F)$ be an algebra and $M\subseteq A\times A$. Then $\langle x, y\rangle\in T(M)$ if and only if there exist a $(k+m+n)$-ary polynomial p over $\mathfrak{A}$ and $x_i, y_i\in A$ $(i=1, \ldots, k+m+n)$ with $x_i=y_i$ for $i\leqq k$, $\langle x_i, y_i\rangle\in M$ for $k<i\leqq k+m$ and*

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$\langle y_i, x_i\rangle \in M$ *for* $k+m<i\leqq k+m+n$ *such that*

$$x = p(x_1, \ldots, x_{k+m+n}), \quad y = p(y_1, \ldots, y_{k+m+n}).$$

Proof. Let R be the set of all $\langle x, y\rangle$ such that there exist a $(k+m+n)$-ary p and x_i, y_i with the prescribed properties. Clearly R is reflexive and symmetric and $M\subseteq R$. The Substitution Property for R can be shown easily by induction on the rank of polynomials, thus $R\in \mathrm{LT}(\mathfrak{A})$ and $T(M)\subseteq R$. If $S\in \mathrm{LT}(\mathfrak{A})$ and $M\subseteq S$ then $\langle x_i, y_i\rangle \in S$ whether $x_i=y_i$, $\langle x_i, y\rangle_i \in M$ or $\langle y_i, x_i\rangle \in M$, hence by the Substitution Property for S, we also have $\langle x, y\rangle \in S$ for $x=p(x_1, \ldots, x_{k+m+n})$, $y=p(y_1, \ldots, y_{k+m+n})$. Hence $R\subseteq S$, implying that $R=T(M)$.

Lemma 2. *Let* $\mathfrak{A}=(A, F)$ *and* $x, y\in A$. *Then* $\langle a, b\rangle \in T(x, y)$ *if and only if there exists a binary algebraic function* φ *over* $\mathfrak{A}$ *such that* $a=\varphi(x, y)$ *and* $b=\varphi(y, x)$.

This follows immediately from Lemma 1.

Theorem. *Let* $\mathscr{V}$ *be a variety of algebras. The following conditions are equivalent:*

(1) $\mathscr{V}$ *is tolerance regular;*

(2) *there exist a* $(3+m+n)$*-ary polynomial* p *and 5-ary polynomials* q_j *such that* $q_j(y, x, x, y, z)=z$ *and*

$$x = p\big(x, y, z, q_1(x, y, x, y, z), \ldots, q_m(x, y, x, y, z), z, \ldots, z\big),$$

$$y = p\big(x, y, z, z, \ldots, z, q_{m+1}(x, y, x, y, z), \ldots, q_{m+n}(x, y, x, y, z)\big).$$

Proof. (1)$\Rightarrow$(2). Let $\mathscr{V}$ be a tolerance regular variety and $\mathfrak{A}=(A, F)= =\mathfrak{F}_3(x, y, z)$ the free algebra of $\mathscr{V}$ with free generators $\{x, y, z\}$. Put $T=T(x, y)$. Since $\mathfrak{A}$ is tolerance regular, we have $T=\mathrm{Tol}\,\{[z]_T\}$, where $[z]_T$ is a tolerance class of T containing z. However, $\mathrm{Tol}\,\{[z]_T\}=T(M)$ for $M=\{z\}\times[z]_T$, thus $T(x, y)=T(M)$. Since $\langle x, y\rangle \in T(M)$, Lemma 1 implies the existence of a polynomial p^* and elements $x_i, y_i\in A$ with

$$x_i = y_i \quad \text{for} \quad i = 1, \ldots, k,$$

$$x_i\in[z]_T, \quad y_i = z \quad \text{for} \quad k < i \leqq k+m,$$

$$x_i = z, \quad y_i\in[z]_T \quad \text{for} \quad k+m < i \leqq k+m+n$$

such that

$$x = p^*(x_1, \ldots, x_{k+m+n}), \quad y = p^*(y_1, \ldots, y_{k+m+n}).$$

Since $\mathfrak{A}=\mathfrak{F}_3(x, y, z)$, there exist ternary polynomials v_i such that $x_i=y_i= =v_i(x, y, z)$ for $i=1, \ldots, k$, i.e. there exists a $(3+m+n)$-ary polynomial p such that

(a)
$$x = p^*(x_1, \ldots, x_{k+m+n}) = p(x, y, z, u_1, \ldots, u_{m+n}),$$

$$y = p^*(y_1, \ldots, y_{k+m+n}) = p(x, y, z, w_1, \ldots, w_{m+n}),$$

where $u_j=x_{j+k}$, $w_j=y_{j+k}$ $(j=1, \dots, m+n)$, i.e.

(b) $$u_j\in[z]_T, \quad w_j = z \quad \text{for} \quad j = 1, \dots, m,$$

$$u_j = z, \quad w_j\in[z]_T \quad \text{for} \quad j = m+1, \dots, m+n.$$

If $u_j\in[z]_T$, $w_j=z$ then $\langle u_j, z\rangle=\langle u_j, w_j\rangle\in \mathrm{Tol}\,\{[z]_T\}=T(x, y)$. By Lemma 2, there exists a binary algebraic function φ_j such that $u_j=\varphi_j(x, y)$ and $z=\varphi_j(y, x)$. Since $\mathfrak{A}=\mathfrak{F}_3(x, y, z)$, there exists a 5-ary polynomial q_j over $\mathcal{V}$ such that $\varphi_j(r, s)=q_j(r, s, x, y, z)$, i.e.

(c) $$u_j = q_j(x, y, x, y, z) \quad \text{and} \quad z = q_j(y, x, x, y, z).$$

We can proceed analogously if $u_j=z$, $w_j\in[z]_T$. Thus (a), (b), (c) imply (2).

(2)$\Rightarrow$(1). Let $\mathfrak{A}=(A, F)\in\mathcal{V}$, $T_1, T_2\in\mathrm{LT}\,(\mathfrak{A})$ and let $[z]_{T_1}=[z]_{T_2}$ be a common tolerance class of T_1, T_2. Suppose $\langle x, y\rangle\in T_1$. Then also

$$\langle q_j(x, y, x, y, z), z\rangle = \langle q_j(x, y, x, y, z),\ q_j(y, x, x, y, z)\rangle\in T_1,$$

i.e. $q_j(x, y, x, y, z)\in[z]_{T_1}=[z]_{T_2}$. Therefore $\langle q_j(x, y, x, y, z), z\rangle\in T_2$ for $j=1, \dots, m+n$. By (2), we have $\langle x, y\rangle\in T_2$, i.e. $T_1\subseteq T_2$. The converse inclusion can be proved analogously, thus $\mathfrak{A}$ and also $\mathcal{V}$ is tolerance regular.

Remark. Since every congruence is a tolerance, tolerance regularity of $\mathcal{V}$ implies regularity of $\mathcal{V}$, i.e. (2) of the theorem is a sufficient condition for the regularity of $\mathcal{V}$. If $\mathcal{V}$ is, moreover, congruence-permutable, then Werner's Theorem in [9] implies $\mathrm{LT}\,(\mathfrak{A})=\mathrm{Con}\,(\mathfrak{A})$ for each $\mathfrak{A}\in\mathcal{V}$, thus (2) of the Theorem is also necessary. A simpler Mal'cev characterization of permutability and regularity is given by the author in [10].

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On sets which contain sum sets

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Introduction. Let $\mathbf{Z}$ be the set of integers and let $E\subset\mathbf{Z}$. For $A\subset\mathbf{Z}$ denote by $|A|$ the cardinal number of A. For $n\geqq 1$, define $\varrho_n(E)=\sup\{\min(|A_1|, |A_2|, \ldots, |A_n|)\colon A_1+A_2+\ldots+A_n\subset E, A_i\subset\mathbf{Z}\}$. For $n\geqq 2$, E is called a ϱ_n set if $\varrho_n(E)<\infty$ and $\varrho_{n-1}(E)=\infty$.

Many authors have studied ϱ_n sets in various connections; see, for example, [1] and [2]. A set D of integers is called *dissociate* if every integer has at most one representation of the form $\varepsilon_1 d_1+\ldots+\varepsilon_m d_m$ with $\varepsilon_j=\pm 1$ $(1\leqq j\leqq m)$ and $d_1<d_2<\ldots<d_m$ are in D. For $n\geqq 1$, suppose D_i, $i=1, 2, \ldots, n$ are such that $D_k\cap D_l=\emptyset$ if $k\neq l$, $|D_i|=\infty$ for $i=1, 2, \ldots, n$, and $\bigcup_{i=1}^{n} D_i$ is a dissociate set. It then follows from [3] that $D_1+D_2+\ldots+D_n$ is a ϱ_{n+1} set which is not a ϱ_n set. The proof of this fact uses the techniques of harmonic analysis. Our purpose here is to indicate, for any $n\geqq 2$, a very simple construction (which uses only the definition of a ϱ_n set) to obtain a new class of sets which are ϱ_{n+1} sets but not ϱ_n sets. We will actually construct a set, $\mathcal{S}$, of positive integers which is the sum of two infinite sets of positive integers such that $\mathcal{S}$ is a ϱ_3 set. It is clear from the construction we give how to construct, for any $n\geqq 2$, a class of sets, $\mathcal{S}_n$, such that $\mathcal{S}_n$ is the sum of n infinite sets of positive integers and $\mathcal{S}_n$ is a ϱ_{n+1} set. Furthermore, it is not hard to see that in our construction $\mathbf{Z}$ may be replaced, with appropriate modifications, by any infinite abelian group; cf. [3].

Moreover, although the sets we construct are not necessarily of the form $D_1+D_2+\ldots+D_n$, where $\bigcup_{i=1}^{n} D_i$ is dissociate, our proof can be easily modified to prove the result for any such sets.

The Construction. Before we begin our construction of the set $\mathcal{S}$, we observe that, since $\mathcal{S}$ will be a set of positive integers, and we will be concerned with showing that $\sup\{\min(|X|, |Y|, |W|)\colon X+Y+W\subset S;\ X, Y, W\subset\mathbf{Z}\}$ is finite, it suffices throughout to consider only sets X, Y, and W of positive integers.

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First take two singletons of positive integers $A_0=\{a_0\}$ and $B_0=\{b_0\}$. Next choose $a_1>3(a_0+b_0)$ and $b_1>\frac{4}{3}a_1$. Write $A_1=A_0\cup\{a_1\}$ and $B_1=B_0\cup\{b_1\}$. In general, if A_n and B_n have already been constructed then take $a_{n+1}>$ $>3\max(A_n+B_n)$ and $b_{n+1}\geqq\frac{4}{3}a_{n+1}$ and write $A_{n+1}=A_n\cup\{a_{n+1}\}$ and $B_{n+1}=$ $=B_n\cup\{b_{n+1}\}$. We define $\mathscr{S}=\bigcup_{n=0}^{\infty}(A_n+B_n)$.

Observe first, that for any n, $A_{n+1}+B_{n+1}=(A_n\cup\{a_{n+1}\})+(B_n\cup\{b_{n+1}\})=$ $=(A_n+B_n)\cup(B_n+\{a_{n+1}\})\cup(A_n+\{b_{n+1}\})\cup\{a_{n+1}+b_{n+1}\}$. Notice that $b_{n+1}>\frac{4}{3}a_{n+1}>$ $>a_{n+1}+\max(A_n+B_n)>\max(a_{n+1}+B_n)$ so that each element of $A_n+\{b_{n+1}\}$ is greater than each element of $B_n+\{a_{n+1}\}$. Also, neither $A_n, B_n, B_n+\{a_{n+1}\}$, nor $A_n+\{b_{n+1}\}$ contains a sum of two doubletons or a translate of a sum of two doubletons. For example, if $x'+(\{y_1,y_2\}+\{w_1,w_2\})\subset a_{n+1}+B_n$ with $y_1<y_2$ and $w_1<w_2$ then for some $b'<b''<b'''<b''''$ in B_n we have

$$x'+y_1+w_1=a_{n+1}+b',\quad x'+y_1+w_2=a_{n+1}+b'',\quad x'+y_2+w_1=a_{n+1}+b''',$$
$$x'+y_2+w_2=a_{n+1}+b'''',$$

or

$$x'+y_1+w_1=a_{n+1}+b',\quad x'+y_2+w_1=a_{n+1}+b'',\quad x'+y_1+w_2=a_{n+1}+b''',$$
$$x'+y_2+w_2=a_{n+1}+b''''.$$

In either case, we obtain $b''''-b'''=b''-b'$ which is impossible since each member of B_n is more than twice its predecessor.

Now suppose that, for some n, if X, Y, and W are sets of positive integers such that if $X+Y+W\subset A_n+B_n$, then $\min(|X|,|Y|,|W|)<4$. Suppose also that X', Y', W' are sets of positive integers with $|X'|$, $|Y'|$, $|W'|$ each at least 4 and $D=X'+Y'+W'\subset A_{n+1}+B_{n+1}$. By the induction hypothesis, $D\not\subset A_n+B_n$ and so $D\cap(B_n+\{a_{n+1}\})\neq\emptyset$ or $D\cap(A_n+\{b_{n+1}\})\neq\emptyset$ or $D\cap\{a_{n+1}+b_{n+1}\}\neq\emptyset$. Thus, some element of X' or Y' or W' must be greater than $\max(A_n+B_n)$ because $3\max(A_n+B_n)<a_{n+1}$. Without loss of generality, call this element $x'\in X'$.

Now, if $|Y'|\geqq 4$ and $|W'|\geqq 4$, then we can see that either $B_n+\{a_{n+1}\}$ or $A_n+\{b_{n+1}\}$ must contain a translate of a sum of two doubletons as follows:

Say $y_1<y_2<y_3<y_4$ are the four smallest elements of Y' and $w_1<w_2<w_3<w_4$ are the four smallest elements of W'. Look at $u_1=x'+y_1+w_1$, $u_2=x'+y_1+w_2$, $u_3=x'+y_2+w_1$, and $u_4=x'+y_2+w_2$. If $u_1\in A_n+\{b_{n+1}\}$, then clearly $u_1,u_2,u_3,$ $u_4\in A_n+\{b_{n+1}\}$. If $u_1\in B_n+\{a_{n+1}\}$, then we're done unless $u_4\in A_n+\{b_{n+1}\}$. But then $x'+(\{y_2,y_3\}+\{w_2,w_3\})\subset A_n+\{b_{n+1}\}$ and we are done. We now have a contradiction and so it follows that $\min(|X'|,|Y'|,|W'|)<4$.

Thus, by induction, for any n, if X, Y, and W are three sets of positive integers with $X+Y+W\subset A_n+B_n$, then $\min(|X|, |Y|, |W|)<4$.

Clearly $\mathscr{S}=\bigcup_{n=0}^{\infty}(A_n+B_n)$ is not ϱ_2. However, if X, Y, and W are three finite sets of positive integers with $X+Y+W\subset\mathscr{S}$, then $X+Y+W\subset A_n+B_n$ for some n and so $\min(|X|, |Y|, |W|)<4$. Thus, $\mathscr{S}$ is a ϱ_3 set. Finally, $\mathscr{S}=\bigcup_{n=0}^{\infty}(A_n+B_n)=\left(\bigcup_{n=0}^{\infty}A_n\right)+\left(\bigcup_{n=0}^{\infty}B_n\right)$ because $A_n\subset A_{n+1}$ and $B_n\subset B_{n+1}$ for all n and so $\mathscr{S}$ is a sum of two infinite sets.

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On the number of prime factors of integers

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1. Throughout this paper, we use the following notations:

$c_1, c_2, \dots$ denote positive absolute constants. The number of elements of a finite set S is denoted by $|S|$. We write $p^\alpha \| n$ if $p^\alpha | n$ but not $p^{\alpha+1} \nmid n$;

$d(n)$ denotes the number of positive divisors of n: $d(n) = \sum_{d|n} 1$;

$\nu(n)$ denotes the number of prime factors of n counted with multiplicities: $\nu(n) = \sum_{p^\alpha \| n} \alpha$;

$\varkappa(n)$ denotes the number of distinct prime factors of n: $\varkappa(n) = \sum_{p|n} 1$;

$\pi_i(x)$ denotes the number of integers n satisfying $n \leqq x$ and $\nu(n) = i$;

$\varrho_i(x)$ denotes the number of integers n satisfying $n \leqq x$ and $\varkappa(n) = i$;

$P(n)$ and $p(n)$ denote the greatest and least prime factor of n, respectively.

2. In [2], the authors asserted that for any $\omega > 0$, there exists a constant $c_1 = c_1(\omega)$ such that for all sufficiently large x and $1 \leqq i \leqq \omega \log\log x$, we have

$$(1) \qquad \pi_i(x) < c_1(\omega) \frac{x}{\log x} \frac{(\log\log x)^{i-1}}{(i-1)!} \quad \text{for} \quad 1 \leqq i \leqq \omega \log\log x.$$

(There was also a missprint: $1 \leqq i \leqq \omega \log x$ was printed instead of $1 \leqq i \leqq \omega \log\log x$.) We attributed this theorem to Hardy and Ramanujan (referring to [4]), and we used it (with $\omega = 100$) to prove that for all $\varepsilon > 0$ and large k,

$$(2) \qquad \sum_{0 \leqq i \leqq z \log\log k} \pi_i(k) < \frac{k}{(\log k)^{\varphi(z) - \varepsilon}}$$

and

$$(3) \qquad \sum_{(1+z)\log\log k < i} \pi_i(k^2) < c_2 \frac{k^2}{(\log k)^{\varphi(z) - \varepsilon}}$$

(see (25) and (33) in [2]) where

$$(4) \qquad \varphi(x) = 1 + x \log x - x$$

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and z denotes the single real root of the equation $\varphi(x)=\varphi(1+x)$; a simple computation shows that

$$0{,}54 < z < 0{,}55. \tag{5}$$

The first author used (1) also in [1], in order to prove that for all $\varepsilon>0$ and $x>x_0(\varepsilon)$, we have

$$\sum_{i>\frac{\log\log x}{\log 2}} \pi_i(x) < \frac{x}{(\log x)^{1-\varepsilon}}(e\log 2)^{\frac{\log\log x}{\log 2}} \tag{6}$$

(see (3) in [1]).

However, (1) is *false* in the form stated above (as K. K. Norton pointed out it in a letter written to the authors). In fact, Hardy and Ramanujan proved (1) with $\varrho_i(x)$ in place of $\pi_i(x)$:

$$\varrho_i(x) < c_3(\omega)\frac{x}{\log x}\frac{(\log\log x)^{i-1}}{(i-1)!} \quad \text{for} \quad 1 \leqq i \leqq \omega\log\log x. \tag{7}$$

Furthermore, they proved in [4] that for all $\delta>0$, (1) holds with $\omega=\frac{10}{9}-\delta$:

$$\pi_i(x) < \frac{c_4}{\delta}\frac{x}{\log x}\frac{(\log\log x)^{i-1}}{(i-1)!} \quad \text{for} \quad 1 \leqq i \leqq \left(\frac{10}{9}-\delta\right)\log\log x.$$

Sathe [6] extended this result by proving that for all $\delta>0$, we have

$$\pi_i(x) < c_5(\delta)\frac{x}{\log x}\frac{(\log\log x)^{i-1}}{(i-1)!} \quad \text{for} \quad x \geqq 3,\ 1 \leqq i \leqq (2-\delta)\log\log x. \tag{8}$$

Selberg [7] gave a different proof of Sathe's result and showed that for all $\delta>0$, we have

$$\pi_i(x) \sim c_6(x\log x)2^{-i} \quad \text{for} \quad (2+\delta)\log\log x \leqq i \leqq c_7\log\log x. \tag{9}$$

This result shows that (1) *does not hold* for $i\geqq(2+\delta)\log\log x$ (while we used (1) with $\omega=100$ in order to prove (3)); in fact, the right hand side of (8) is greater than the right hand side of (1). (See also [3] and [5].)

The aim of this paper is to correct the papers [1] and [2] by deducing an upper estimate for $\pi_i(x)$ which is slightly weaker than the best possible but which holds for *all* i:

Theorem 1. *For all $\delta>0$, we have*

$$\pi_i(x) < \begin{cases} c_8(\delta)\dfrac{x}{\log x}\dfrac{(\log\log x)^{i-1}}{(i-1)!} & \textit{for} \quad 1 \leqq i \leqq (2-\delta)\log\log x \\ c_9 i^4\dfrac{x\log x}{2^i} & \textit{for} \quad 1 \leqq i \end{cases} \tag{10}$$

and for all $x\geqq 3$.

Sections 3 and 4 will be devoted to the proof of this theorem. In Section 5, we prove two corollaries of Theorem 1. In Section 6, we show that in fact, (2), (3) and (6) can be deduced easily from these corollaries.

3. In order to prove Theorem 1, we need two lemmas.

Lemma 1. *For all non-negative real numbers Z and A, let $G(Z, A)$ denote the number of positive integers n satisfying $n \leqq Z$ and $\varkappa(n) \geqq A$. Then there exists an absolute constant c_{10} such that for all Z and A, we have*

$$G(Z, A) \leqq c_{10} 2^{-A} Z \log (Z+2). \tag{11}$$

Proof. If $\varkappa(n) \geqq A$ then we have

$$d(n) = \prod_{p^\alpha \| n} d(p^\alpha) \geqq \prod_{p^\alpha \| n} 2 = \prod_{p | n} 2 = 2^{\varkappa(n)} \geqq 2^A$$

thus

$$\sum_{n \leqq Z} d(n) \geqq \sum_{\substack{n \leqq Z \\ \varkappa(n) \geqq A}} d(n) \geqq \sum_{\substack{n \leqq Z \\ \varkappa(n) \geqq A}} 2^A = 2^A G(Z, A). \tag{12}$$

On the other hand, it is well-known that for $Z \to +\infty$,

$$\sum_{n \leqq Z} d(n) \sim Z \log Z$$

thus for all $Z (\geqq 0)$, we have

$$\sum_{n \leqq Z} d(n) \leqq c_{11} Z \log (Z+2). \tag{13}$$

(12) and (13) yield (11).

Lemma 2. *For a positive real number y and a non-negative integer α, write*

$$F(y, \alpha) = \sum_{\substack{P(n) \leqq y \\ \nu(n) = \alpha}} \frac{1}{n}.$$

Then there exists an absolute constant c_{12} such that for $y \geqq 2$ and all α, we have

$$F(y, \alpha) \leqq c_{12} (\alpha+1) 2^{-\alpha} (\log y)^2. \tag{14}$$

Proof. Let us write

$$f(t) = \prod_{p \leqq y} \sum_{k=0}^{\alpha} \left(\frac{t}{p}\right)^k = \sum_{i=0}^{m} a_i t^i$$

(where $m = \alpha \pi(y)$). Then obviously, all the coefficients a_i are non-negative and we have $F(y, \alpha) = a_\alpha$. Thus

$$f(2) = \sum_{i=0}^{m} a_i 2^i \geqq a_\alpha 2^\alpha = 2^\alpha F(y, \alpha). \tag{15}$$

On the other hand, by the definition of $f(t)$ and using the Mertens-formula, we obtain that

$$(16)\quad f(2)=\prod_{p\le y}\sum_{k=0}^{\alpha}\left(\frac{2}{p}\right)^k=(\alpha+1)\prod_{3\le p\le y}\sum_{k=0}^{\alpha}\left(\frac{2}{p}\right)^k\le(\alpha+1)\prod_{3\le p\le y}\sum_{k=0}^{+\infty}\left(\frac{2}{p}\right)^k=$$

$$=(\alpha+1)\prod_{3\le p\le y}\frac{1}{1-\frac{2}{p}}=(\alpha+1)\left(\prod_{3\le p\le y}\frac{1}{1-\frac{1}{p}}\right)^2\prod_{3\le p\le y}\left(1-\frac{1}{p}\right)^2\left(1-\frac{2}{p}\right)^{-1}=$$

$$=(\alpha+1)\left(\prod_{3\le p\le y}\frac{1}{1-\frac{1}{p}}\right)^2\prod_{3\le p\le y}\frac{(p-1)^2}{(p-2)p}<(\alpha+1)\left(\prod_{p\le y}\frac{1}{1-\frac{1}{p}}\right)^2\prod_{n=3}^{+\infty}\frac{(n-1)^2}{(n-2)n}<$$

$$<(\alpha+1)(c_{13}\log y)^2\cdot 2=c_{14}(\alpha+1)(\log y)^2.$$

(15) and (16) yield (14).

4. Completion of the proof of Theorem 1. If $1\le i\le(2-\delta)\log\log x$ then the first inequality in (10) holds by the Sathe—Selberg formula (8). Thus it is sufficient to prove that

$$(17)\qquad \pi_i(x)<c_9 i^4\frac{x\log x}{2^i}\quad\text{for all}\quad x\ge 3\ \text{and}\ 1\le i.$$

Let us fix a real number $x\ge 3$ and a positive integer i. Let S denote the set of the positive integers n satisfying $n\le x$ and $v(n)=i$ (so that $\pi_i(x)=|S|$). Furthermore, let S_1 denote the set of the positive integers n for which $n\le x$ and there exists a positive integer t such that $t>2^i$ and t^2/n. Write $S_2=S-S_1$. Then we have

$$(18)\qquad S\subset S_1\cup S_2$$

and

$$(19)\qquad S_1\cap S_2=\emptyset.$$

(18) implies that

$$(20)\qquad \pi_i(x)=|S|\le|S_1|+|S_2|.$$

Obviously, we have

$$(21)\qquad |S_1|=\sum_{t=2^i+1}^{+\infty}\sum_{\substack{n\le x\\ t^2|n}}1=\sum_{t=2^i+1}^{+\infty}\left[\frac{x}{t^2}\right]<x\sum_{t=2^i+1}^{+\infty}\frac{1}{t^2}<$$

$$<x\sum_{t=2^i+1}^{+\infty}\frac{1}{(t-1)t}=x\sum_{t=2^i+1}^{+\infty}\left(\frac{1}{t-1}-\frac{1}{t}\right)=\frac{x}{2^i}.$$

In order to estimate $|S_2|$, let us write all $n \in S$ in the form $n = n_1 n_2$ where

(22) $$P(n_1) \leqq 2^i, \quad p(n_2) > 2^i.$$

If there exists a prime number p such that $p > 2^i$ and $p^2 | n_2$ then by the definition of S_1, we have $n \in S_1$ thus by (19), $n \notin S_2$. In other words, for all $n \in S_2$, n_2 is squarefree thus

(23) $$\varkappa(n_2) = \nu(n_2) = \nu(n) - \nu(n_1) = i - \nu(n_1).$$

If $n \in S_2$ and we put $\nu(n_1) = \alpha$ then by (23),

(24) $$0 \leqq \alpha = i - \varkappa(n_2) \leqq i.$$

By (22), (23) and (24), we have

$$|S_2| = \sum_{\substack{n_1 n_2 \leqq x \\ P(n_1) \leqq 2^i,\ p(n_2) > 2^i \\ \nu(n_1) + \varkappa(n_2) = i}} 1 = \sum_{\alpha=0}^{i} \sum_{\substack{n_1 \leqq x \\ P(n_1) \leqq 2^i \\ \nu(n_1) = \alpha}} \sum_{\substack{n_2 \leqq x/n_1 \\ p(n_2) > 2^i \\ \varkappa(n_2) = i-\alpha}} 1 \leqq \sum_{\alpha=0}^{i} \sum_{\substack{n_1 = x \\ P(n_1) \leqq 2^i \\ \nu(n_1) = \alpha}} \sum_{\substack{n_2 \leqq x/n_1 \\ \varkappa(n_2) \geqq i-\alpha}} 1.$$

In order to estimate the inner sum, we use Lemma 1 with $Z = x/n_1$ and $A = i - \alpha$ We obtain that

$$|S_2| \leqq \sum_{\alpha=0}^{i} \sum_{\substack{n_1 \leqq x \\ P(n_1) \leqq 2^i \\ \nu(n_1) = \alpha}} c_{10} 2^{-i+\alpha} \frac{x}{n_1} \log\left(\frac{x}{n_1} + 2\right) <$$

$$< c_{10} \sum_{\alpha=0}^{i} \sum_{\substack{n_1 \leqq x \\ P(n_1) \leqq 2^i \\ \nu(n_1) = \alpha}} 2^{-i+\alpha} \frac{x}{n_1} \log(x+2) <$$

$$< c_{15} \sum_{\alpha=0}^{i} 2^{-i+\alpha} x \log x \sum_{\substack{P(n_1) \leqq 2^i \\ \nu(n_1) = \alpha}} \frac{1}{n_1} = c_{15} \sum_{\alpha=0}^{i} 2^{-i+\alpha} x \log x F(2^i, \alpha)$$

where $F(y, \alpha)$ is defined in Lemma 2. By using Lemma 2, we obtain that

(25) $$|S_2| < c_{15} \sum_{\alpha=0}^{i} 2^{-i+\alpha} x \log x \cdot c_{12} (\alpha+1) 2^{-\alpha} (\log 2^i)^2 <$$

$$< c_{16} i^2 2^{-i} x \log x \sum_{\alpha=0}^{i} (\alpha+1) < c_{17} i^4 2^{-i} x \log x.$$

(20), (21) and (25) yield that

$$\pi_i(x) \leqq |S_1| + |S_2| < 2^{-i} x + c_{17} i^4 2^{-i} x \log x <$$

$$< c_{18} i^4 2^{-i} x \log x$$

which proves (17) and this completes the proof of Theorem 1.

5. It can be deduced easily from Theorem 1 that

Corollary 1. *If*

$$\delta > 0 \quad \text{and} \quad 1 < y < 2-\delta \tag{26}$$

then we have

$$\sum_{i \geqq j} \pi_i(x) < c_{19}(\delta) \frac{1}{y-1} \frac{x}{\log x} \frac{(\log\log x)^{j-1}}{(j-1)!} \tag{27}$$

$$\text{for} \quad y \leqq j/\log\log x \leqq 2-\delta, \quad x > x_0(y, \delta);$$

furthermore, we have

$$\sum_{i \geqq j} \pi_i(x) < c_{20}\, j^4 \frac{x \log x}{2^j} \quad \text{for all } j \text{ and } \quad x \geqq 3. \tag{28}$$

Proof. First we prove (28). By Theorem 1, we have

$$\sum_{i \geqq j} \pi_i(x) < \sum_{i \geqq j} c_9\, i^4 \frac{x \log x}{2^i} = c_9 x \log x \sum_{i \geqq j} \frac{i^4}{2^i}. \tag{29}$$

Obviously, for $i > i_0$ we have

$$\frac{(i+1)^4}{2^{i+1}} < \frac{2}{3} \frac{i^4}{2^i}$$

thus for $j \geqq i_0$,

$$\sum_{i \geqq j} \frac{i^4}{2^i} < \frac{j^4}{2^j} \sum_{t=0}^{+\infty} \left(\frac{2}{3}\right)^t = 3 \cdot \frac{j^4}{2^j}$$

hence

$$\sum_{i \geqq j} \frac{i^4}{2^i} < \max\left\{\sum_{i=1}^{+\infty} \frac{i^4}{2^i} \left(\max_{1 \leqq t \leqq i_0} 2^t t^{-4}\right), 3\right\} \cdot \frac{j^4}{2^j} = c_{21} \frac{j^4}{2^j} \tag{30}$$

for all j. (29) and (30) yield (28).

Now we prove (27). The function $\varphi(x) = 1 + x\log x - x$ is increasing for $x > 1$, thus writing

$$\eta = \eta(\delta) = \frac{\varphi(2) - \varphi(2-\delta)}{2\log 2},$$

we have $0 < \eta$. Thus Theorem 1 and (28) yield (with respect to (26)) that for

$x > x_0(y, \delta)$,

(31)
$$\sum_{i \geqq j} \pi_i(x) = \sum_{j \leqq i \leqq [(2-\eta)\log\log x]} \pi_i(x) + \sum_{[(2-\eta)\log\log x]+1 \leqq i} \pi_i(x) <$$

$$< \sum_{j \leqq i \leqq [(2-\eta)\log\log x]} c_8(\eta(\delta)) \frac{x}{\log x} \frac{(\log\log x)^{i-1}}{(i-1)!} +$$

$$+ c_{20} \frac{([(2-\eta)\log\log x]+1)^4}{2^{[(2-\eta)\log\log x]+1}} x \log x <$$

$$< c_{22}(\delta) \frac{x}{\log x} \frac{(\log\log x)^{j-1}}{(j-1)!} \sum_{j \leqq i \leqq [(2-\eta)\log\log x]} \frac{(\log\log x)^{i-j}}{j(j+1)\dots(i-1)} +$$

$$+ c_{21} \frac{(2\log\log x)^4}{2^{(2-\eta)\log\log x}} x \log x <$$

$$< c_{22}(\delta) \frac{x}{\log x} \frac{(\log\log x)^{j-1}}{(j-1)!} \sum_{j \leqq i} \left(\frac{\log\log x}{j}\right)^{i-j} +$$

$$+ c_{23} x \frac{(\log\log x)^4}{(\log x)^{(2-2\eta)\log 2 + \eta \log 2 - 1}} <$$

$$< c_{22}(\delta) \frac{x}{\log x} \frac{(\log\log x)^{j-1}}{(j-1)!} \sum_{t=0}^{+\infty} y^{-t} + \frac{x}{(\log x)^{(2-2\eta)\log 2 - 1 + \eta/2}} =$$

$$= c_{22}(\delta) \frac{y}{y-1} \frac{x}{\log x} \frac{(\log\log x)^{j-1}}{(j-1)!} + \frac{x}{(\log x)^{\varphi(2-\delta)+\eta/2}} <$$

$$< c_{23}(\delta) \frac{1}{y-1} \frac{x}{\log x} \frac{(\log\log x)^{j-1}}{(j-1)!} + \frac{x}{(\log x)^{\varphi(2-\delta)+\eta/2}}.$$

By the Stirling-formula, we have

(32)
$$\frac{1}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!} = \frac{1}{\log x} \frac{k}{\log\log x} \frac{(\log\log x)^k}{k!} \sim$$

$$\sim c_{24} \frac{1}{\log x} \frac{k}{\log\log x} \left(\frac{e \log\log x}{k}\right)^k k^{-1/2} =$$

$$= c_{24} \frac{k^{1/2}}{\log\log x} (\log x)^{-1+(1-\log(k/\log\log x))k/\log\log x} =$$

$$= c_{24} \frac{k^{1/2}}{\log\log x} (\log x)^{-\varphi(k/\log\log x)} \quad \text{for} \quad x \geqq 3 \text{ and } k \to +\infty.$$

Thus with respect to (26), for $y \leqq j/\log\log x \leqq 2-\delta$, $x > x_1(y, \delta, \eta) = x_1(y, \delta, \eta(\delta)) = x_2(y, \delta)$ we have

$$c_{23}(\delta)\frac{1}{y-1}\frac{x}{\log x}\frac{(\log\log x)^{j-1}}{(j-1)!} > \tag{33}$$

$$> c_{25}(y, \delta)x(\log\log x)^{-1/2}(\log x)^{-\varphi(j/\log\log x)} >$$

$$> c_{25}(y, \delta)x(\log\log x)^{-1/2}(\log x)^{-\varphi(2-\delta)} > x(\log x)^{-\varphi(2-\delta)-\eta/2}.$$

(31) and (33) yield (27) and this completes the proof of Corollary 1.

Corollary 2. *If* $y>1$ *and* $\varepsilon>0$ *then for* $y\log\log x \leqq j$, $x > x_0(\varepsilon)$ *we have*

$$\sum_{i\geqq j}\pi_i(x) < \begin{cases} \dfrac{x}{(\log x)^{\varphi(y)-\varepsilon}} & \textit{if } 1 < y < 2 \\ \dfrac{x}{(\log x)^{(1-\varepsilon)y\log 2-1}} & \textit{if } 2 \leqq y. \end{cases} \tag{34}$$

Proof. If $1+(\log x)^{-\varepsilon/2} < y \leqq 2-\varepsilon/2$ then (27) (with $\varepsilon/2$ in place of δ) and (32) yield that

$$\sum_{i\geqq j}\pi_i(x) < c_{19}(\varepsilon/2)\frac{1}{y-1}x\frac{1}{\log x}\frac{(\log\log x)^{j-1}}{(j-1)!} <$$

$$< c_{26}(\varepsilon)(\log x)^{\varepsilon/2}x\frac{j^{1/2}}{\log\log x}(\log x)^{-\varphi(j/\log\log x)} < x(\log x)^{-\varphi(y)+\varepsilon}$$

for $x > x_1(\varepsilon)$, while if $1 < y \leqq 1+(\log x)^{-\varepsilon/2}$ then (34) holds trivially for $x > x_2(\varepsilon)$ since we have

$$\lim_{t\to 1}\varphi(t) = \varphi(1) = 0$$

and, for all j,

$$\sum_{i\geqq j}\pi_i(x) \leqq x.$$

If $2-\varepsilon/2 < y$ then by (28), we have

$$\sum_{i\geqq j}\pi_i(x) < c_{20}j^4\frac{x\log x}{2^j} < \frac{x\log x}{2^{(1-\varepsilon/4)j}} \leqq \tag{35}$$

$$\leqq \frac{x\log x}{2^{(1-\varepsilon/4)y\log\log x}} = \frac{x}{(\log x)^{(1-\varepsilon/4)y\log 2-1}}$$

for $x>x_3(\varepsilon)$. If $2\leqq y$ then this yields (34). Finally, if $2-\varepsilon/2<y\leqq 2$ then we obtain from (35) that

$$\sum_{i\geqq j}\pi_i(x)<\frac{x}{(\log x)^{(1-\varepsilon/4)y\log 2-1}}=$$

$$=\frac{x}{(\log x)^{(1+y\log y-y)+y(\log 2-\log y)-(2-y)-(\varepsilon y\log 2)/4}}=$$

$$=\frac{x}{(\log x)^{\varphi(y)+y(\log 2-\log y)-(2-y)-(\varepsilon y\log 2)/4}}<$$

$$<\frac{x}{(\log x)^{\varphi(y)-\varepsilon/2-\varepsilon/2}}=\frac{x}{(\log x)^{\varphi(y)-\varepsilon}}$$

which completes the proof of (34).

6. In this section, we correct the proofs of (2), (3) and (6). In the proof of (2), we used (1) only for $i\leqq z\log\log k$. Thus we need (1) with $\omega=z<0{,}55<10/9$ but in this case, (1) holds by the classical Hardy—Ramanujan result.

Now we are going to prove (3). Let $\delta=\delta(\varepsilon)$ denote a small positive number such that we have

$$\varphi(1+z-\delta)>\varphi(1+z)-\varepsilon/2=\varphi(z)-\varepsilon/2$$

(note that $\varphi(1+z)=\varphi(z)$ by the definition of z). By using Corollary 2 with $1+z-\delta$, $\varepsilon/2$, k^2 and $[(1+z-\delta)\log\log k^2]+1$ in place of y, ε, x and j, respectively, we obtain that

$$\sum_{(1+z)\log\log k<i}\pi_i(k^2)<\sum_{[(1+z-\delta)\log\log k^2]+1\leqq i}\pi_i(k^2)<$$

$$<\frac{k^2}{(\log k^2)^{\varphi(1+z-\delta)-\varepsilon/2}}<\frac{k^2}{(\log k)^{\varphi(z)-\varepsilon/2-\varepsilon/2}}=\frac{k^2}{(\log k)^{\varphi(z)-\varepsilon}}$$

for $k>k_0(\varepsilon)$ which proves (3).

Finally, note that the right hand side of (6) can be rewritten in the form

$$\frac{x}{(\log x)^{1-\varepsilon}}(e\log 2)^{\frac{\log\log x}{\log 2}}=\frac{x}{(\log x)^{1-\varepsilon-(1+\log\log 2)/\log 2}}=\frac{x}{(\log x)^{\varphi(1/\log 2)-\varepsilon}}$$

so that (6) can be obtained from Corollary 2 with $1/\log 2$ (<2) in place of y.

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A Hausdorff—Young type inequality and necessary multiplier conditions for Jacobi expansions

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1. Introduction. We shall show how necessary conditions for Jacobi multipliers can be derived from certain Hausdorff—Young type inequalities.

In order to become more precise we first have to introduce the following notation. Fix $\alpha\geqq\beta\geqq-\frac{1}{2}$ and let $L^p_{(\sigma,\tau)}=L^p_{(\sigma,\tau;\alpha,\beta)}$, $1\leqq p<\infty$, denote the space of measurable functions on $[0,\pi]$ such that

$$\|f\|_{p;\sigma,\tau}=\left(\int_0^\pi\left|\left(\sin\frac{\theta}{2}\right)^\sigma\left(\cos\frac{\theta}{2}\right)^\tau f(\theta)\right|^p d\mu(\theta)\right)^{1/p}$$

is finite where

$$d\mu(\theta)=d\mu^{(\alpha,\beta)}(\theta)=\left(\sin\frac{\theta}{2}\right)^{2\alpha+1}\left(\cos\frac{\theta}{2}\right)^{2\beta+1}d\theta.$$

If $\tau=0$ we write $L^p_{(\sigma,0)}=L^p_\sigma$, $\|\cdot\|_{p;\sigma,0}=\|\cdot\|_{p,\sigma}$ and if, additionally, $\sigma=0$ we use the standard notations L^p, $\|\cdot\|_p$. Note that $L^\infty\subset L^p_{(\sigma,\tau)}\subset L^1$ if $-(2\alpha+2)<\sigma p<(2\alpha+2)(p-1)$, $-(2\beta+2)<\tau p<(2\beta+2)(p-1)$. Here, as elsewhere, the inclusion sign means that the identity map is continuous. Each $f\in L^1$ has an expansion of the form

$$f(\theta)\sim\sum_{k=0}^\infty f^\wedge(k)\,h_k R_k(\cos\theta)$$

where $R_k(x)=R_k^{(\alpha,\beta)}(x)=P_k^{(\alpha,\beta)}(x)/P_k^{(\alpha,\beta)}(1)$, $P_k^{(\alpha,\beta)}(x)$ being the Jacobi polynomial of degree k and order (α,β), [8]. Also the k-th Fourier—Jacobi coefficient $f^\wedge(k)$ is defined by

$$f^\wedge(k)=\int_0^\pi f(\theta)R_k(\cos\theta)\,d\mu(\theta)$$

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and $h_k = h_k^{(\alpha,\beta)} = \|R_k(\cos\theta)\|_2^{-2} \approx k^{2\alpha+1}$, where the $\approx$ sign means that there are positive constants C, C' such that $C'h_k \leqq k^{2\alpha+1} \leqq Ch_k$ ($k^{2\alpha+1}$ must be replaced by 1 when $k=0$).

A sequence $m=\{m_k\}_0^\infty \in l^\infty$ is called a multiplier from $L^p_{(\sigma,\tau)}$ into $L^q_{(\sigma,\tau)}$, notation $m \in M^q_p(\sigma,\tau) = M^q_p(\sigma,\tau;\alpha,\beta)$, if for each $f \in L^p_{(\sigma,\tau)}$ there exists a function $Mf \in L^q_{(\sigma,\tau)}$ with

$$(1.1) \qquad Mf(\theta) \sim \sum_{k=0}^{\infty} m_k f^\wedge(k) h_k R_k(\cos\theta), \quad \|Mf\|_{q;\sigma,\tau} \leqq C\|f\|_{p;\sigma,\tau}.$$

The smallest constant C independent of f for which this holds is called the multiplier norm of m and it is denoted by $\|m\|_{M^q_p(\sigma,\tau)}$. If $\tau=0$ we write $M^q_p(\sigma,0) = M^q_p(\sigma)$.

The derivation of sharp sufficient multiplier conditions (see e.g. [2], [3], [6]) relies heavily upon the following Parseval type inequality ($f(\theta)$ being a polynomial in $\cos\theta$)

$$(1.2) \qquad \left\|\left(\sin\frac{\theta}{2}\right)^{\gamma} f(\theta)\right\|_2^2 \leqq C \sum_{k=0}^{\infty} |\Delta^\gamma f^\wedge(k)|^2 h_k, \quad -\frac{1}{2} < \gamma < \alpha+2,$$

where the fractional difference operator Δ^γ, $\gamma \in R$, is defined by

$$\Delta^\gamma m_k = \sum_{j=k}^{\infty} A_{j-k}^{-\gamma-1} m_j, \quad A_k^\gamma = \binom{k+\gamma}{k} = \frac{\Gamma(k+\gamma+1)}{\Gamma(k+1)\Gamma(\gamma+1)},$$

whenever the series converges. So one can expect that the converse of (1.2)

$$(1.3) \qquad \sum_{k=0}^{\infty} |\Delta^\gamma f^\wedge(k)|^2 h_k \leqq C \left\|\left(\sin\frac{\theta}{2}\right)^{\gamma} f(\theta)\right\|_2^2, \quad \gamma > -1,$$

proved *in* [7] *for functions* $f(\theta)$ *which are polynomials in* $\cos\theta$, *will yield necessary* multiplier conditions; this will turn out to be true on L^2_σ. However, to obtain necessary conditions also on L^p_σ, $p \neq 2$, we shall need a Hausdorff—Young type variant of (1.3).

The plan of this paper is as follows. In Sec. 2 we derive for the special case $\alpha = \beta = -\frac{1}{2}$, i.e. for cosine expansions, the desired Hausdorff—Young type inequality and deduce from it necessary multiplier conditions on L^p_σ. Then in Sec. 3 we consider the general case $\alpha \geqq \beta \geqq -\frac{1}{2}$, $\alpha > -\frac{1}{2}$, and derive the corresponding Hausdorff—Young type inequality and necessary multiplier conditions. Finally we close with several remarks concerning our results.

2. Necessary multiplier conditions for cosine expansions in weighted Lebesgue spaces. Consider $f \in L^1$ and observe that, since $R_k^{(-1/2,-1/2)}(\cos\theta) = \cos k\theta$,

$$f^\wedge(k) = \int_0^\pi f(\theta)\cos k\theta \, d\theta = \frac{1}{2}\int_0^\pi f(\theta)(e^{ik\theta}+e^{-ik\theta})\, d\theta$$

and hence

$$\Delta^\gamma f\,\hat{}\,(k) = \frac{1}{2}\int_0^\pi f(\theta) \sum_{j=k}^\infty A_{j-k}^{-\gamma-1}(e^{ij\theta}+e^{-ij\theta})\,d\theta =$$

$$= \frac{1}{2}\int_0^\pi f(\theta)\{e^{ik\theta}(1-e^{i\theta})^\gamma + e^{-ik\theta}(1-e^{-i\theta})^\gamma\}\,d\theta.$$

Thus we obtain

$$\sup_k |\Delta^\gamma f\,\hat{}\,(k)| \leqq C\int_0^\pi \left|\left(\sin\frac{\theta}{2}\right)^\gamma f(\theta)\right| d\theta \tag{2.1}$$

and hence, by applying the Riesz—Thorin interpolation theorem to (1.3) and (2.1)

$$\left(\sum_{k=0}^\infty |\Delta^\gamma f\,\hat{}\,(k)|^{p'}\right)^{1/p'} \leqq C\|f\|_{p,\gamma}, \quad 1 \leqq p \leqq 2,\ \gamma \geqq 0, \tag{2.2}$$

for polynomials $f(\theta)$ in $\cos\theta$, where p' is defined by $\frac{1}{p}+\frac{1}{p'}=1$. To state our necessary conditions for cosine multipliers we need to use the following sequence spaces of weak bounded variation (see [4]): for $1 \leqq q \leqq \infty$, $\gamma > 0$,

$$wbv_{q,\gamma} = \{m \in l^\infty\colon\ \|m\|_{q,\gamma;w} < \infty\}$$

where

$$\|m\|_{q,\gamma;w} = \|m\|_\infty + \sup_{j\in N}\left(\sum_{k=2^{j-1}}^{2^j-1} k^{-1}|k^\gamma \Delta^\gamma m_k|^q\right)^{1/q}$$

for $q<\infty$ and, in case $q=\infty$,

$$\|m\|_{\infty,\gamma;w} = \|m\|_\infty + \sup_{k\in N}|k^\gamma \Delta^\gamma m_k|.$$

Theorem 1. *If* $0<\gamma<1-1/p$, $1<p\leqq 2$, *and* $m\in M_p^p(\gamma)$, *then* $m\in wbv_{p',\gamma}$ *and* $\|m\|_{p',\gamma;w} \leqq C\|m\|_{M_p^p(\gamma)}$, *i.e.*, $M_p^p(\gamma) \subset wbv_{p',\gamma}$.

Remark. At the AMS Summer Institute in Williamstown, Mass., 1978, Muckenhoupt, Wheeden and Wo-Sang Young announced necessary and sufficient conditions for a sequence to belong to $M_2^2(\gamma)$ when $\gamma>1/2$. Here we treat the case $-\frac{1}{2}<\gamma<\frac{1}{2}$, $\gamma\neq 0$ (note: $M_2^2(\gamma)=M_2^2(-\gamma)$). By combining the sufficient condition in [6] and the present necessary one it follows for $0<|\gamma|<\frac{1}{2}$ that

$$wbv_{q,\delta} \subset M_2^2(\gamma) \subset wbv_{2,|\gamma|}, \quad \delta > \max\left\{\frac{1}{q},\ |\gamma|\right\}.$$

Proof of Theorem 1. For the Dirichlet kernel

$$D_n(\theta) = 1+2\sum_{k=1}^{n}\cos k\theta = \frac{\sin\left(n+\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}},$$

which we will use as a testfunction, it is easy to check that $\|D_n\|_{p,\gamma} \leqq Cn^{1-\frac{1}{p}-\gamma}$, $-\frac{1}{p}<\gamma<1-\frac{1}{p}$. In order to apply our Hausdorff—Young type inequality (2.2) we need a decomposition of $\{m_k\}$ into a set of sequences with finite support. In fact, setting

$$E_k(l) = \begin{cases} 1, & 2^{l-1} \leqq k < 2^l, \\ 0, & \text{otherwise}, \end{cases}$$

each sequence $E(l)=\{E_k(l)\}_{k=0}^{\infty}$ has support on a dyadic block and so by (2.2)

$$\left(\sum_{2^{j-1}}^{2^j-1} |\Delta^\gamma m_k|^{p'}\right)^{1/p'} \leqq \tag{2.3}$$

$$\leqq \left(\sum_{k=2^{j-1}}^{2^j-1} |\Delta^\gamma(m_k(E_k(j)+E_k(j+1)))|^{p'}\right)^{1/p'} + \sum_{l=j+1}^{\infty}\left(\sum_{k=2^{j-1}}^{2^j-1} |\Delta^\gamma(m_k E_k(l))|^{p'}\right)^{1/p'} \leqq$$

$$\leqq C\|m\|_{M_p^p(\gamma)}\|D_{2^{j+1}-1}-D_{2^{j-1}-1}\|_{p,\gamma}+\|m\|_\infty \sum_{l=j+1}^{\infty}\left(\sum_{k=2^{j-1}}^{2^j-1}(2^l)^{-\gamma p'}\right)^{1/p'}$$

since for $2^{j-1} \leqq k < 2^j$, $l \geqq j+1$,

$$\Delta^\gamma(m_k E_k(l)) = \sum_{n=2^{l-1}}^{2^l-1} A_{n-k}^{-1-\gamma} m_n \leqq C\|m\|_\infty (2^l)^{-\gamma}.$$

Now observe that $\|m\|_\infty \leqq C\|m\|_{M_p^p(\gamma)}$ and multiply both sides of (2.3) by $(2^j)^{\gamma+\frac{1}{p}-1}$ to obtain

$$\left(\sum_{2^{j-1}}^{2^j-1} k^{-1}|k^\gamma \Delta^\gamma m_k|^{p'}\right)^{1/p'} \leqq C\|m\|_{M_p^p(\gamma)}\left\{1+(2^j)^\gamma \sum_{l=j+1}^{\infty}(2^l)^{-\gamma}\right\} \leqq C\|m\|_{M_p^p(\gamma)},$$

which establishes the theorem since C is independent of j.

3. The case $\alpha \geqq \beta \geqq -\frac{1}{2}, \alpha > -\frac{1}{2}$. In [7, Sec. 2] it was shown that if $\gamma > -1$, $f(\theta)$ is a polynomial in $\cos\theta$, and

$$d_k = \int_0^\pi f(\theta) R_k^{(\alpha+\gamma,\beta)}(\cos\theta)\left(\sin\frac{\theta}{2}\right)^{2\alpha+2\gamma+1}\left(\cos\frac{\theta}{2}\right)^{2\beta+1} d\theta$$

then

$$\Delta^\gamma f^\wedge(n) = \sum_{k=n}^{\infty} B_k(n) d_k \tag{3.1}$$

where $B_k(n)=B_k(n;\alpha,\beta,\gamma)=O(k^{\gamma-1})$ for $k\geqq n+1$ and $B_n(n)=O(n^\gamma)$. Since, by SZEGŐ [8; Theorem 7.32.3],

$$\left|\sqrt{h_k^{(\alpha,\beta)}}\,R_k^{(\alpha,\beta)}(\cos\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+\frac12}\left(\cos\frac{\theta}{2}\right)^{\beta+\frac12}\right|\leqq C$$

we have

$$d_k\sqrt{h_k^{(\alpha+\gamma,\beta)}}\leqq C\int_0^\pi |f(\theta)|\left(\sin\frac{\theta}{2}\right)^{\alpha+\gamma+\frac12}\left(\cos\frac{\theta}{2}\right)^{\beta+\frac12}d\theta$$

and so from (3.1) and the fact that $h_k^{(\alpha+\gamma,\beta)}\approx k^{2\alpha+2\gamma+1}$ it follows that

$$\sup_n\left|\sqrt{h_n}\,\Delta^\gamma f^\wedge(n)\right|\leqq C\int_0^\pi\left|\left(\sin\frac{\theta}{2}\right)^{\alpha+\gamma+\frac12}\left(\cos\frac{\theta}{2}\right)^{\beta+\frac12}f(\theta)\right|d\theta. \tag{3.2}$$

Application of the Riesz—Thorin theorem to (3.2) and our previous result [7; Theorem 1b]

$$\left(\sum_{n=0}^\infty\left|\sqrt{h_n}\,\Delta^\gamma f^\wedge(n)\right|^2\right)^{1/2}\leqq C\left(\int_0^\pi\left|\left(\sin\frac{\theta}{2}\right)^{\alpha+\gamma+\frac12}\left(\cos\frac{\theta}{2}\right)^{\beta+\frac12}f(\theta)\right|^2d\theta\right)^{\frac12} \tag{3.3}$$

then gives (in combination with (2.2))

Theorem 2. *Let* $1\leqq p\leqq 2,\ \gamma\geqq 0,\ \alpha\geqq\beta\geqq-\frac{1}{2},\ \alpha>-\frac{1}{2}$ *and let* $f(\theta)$ *be a polynomial in* $\cos\theta$. *Then there exists a constant* C *independent of* f *such that*

$$\left(\sum_{n=0}^\infty\left|\sqrt{h_n}\,\Delta^\gamma f^\wedge(n)\right|^{p'}\right)^{1/p'}\leqq C\|f\|_{p;\sigma,\tau}$$

where $\sigma=\gamma+(2\alpha+1)\left(\frac{1}{2}-\frac{1}{p}\right)$ *and* $\tau=(2\beta+1)\left(\frac{1}{2}-\frac{1}{p}\right)$.

Unfortunately, when $\alpha>-\frac{1}{2}$ the Dirichlet kernel for Jacobi series is too bad a test function to obtain necessary conditions for multipliers on $L^p_{(\sigma,\tau;\alpha,\beta)}$ analogous to Theorem 1. In order to estimate test functions with "nice" Jacobi coefficients we shall need

Lemma 1. *Let* $\alpha,\beta,p,\sigma,\tau$ *be as in Theorem 2 and let* $\{g_k\}_{k=0}^\infty\in l^\infty$ *have compact support*; *set*

$$I_\lambda:=\sum_{k=0}^\infty k^{\lambda+\alpha+\frac32-\gamma-\frac1p}\left|\Delta^{\lambda+1}g_k\right|. \tag{3.4}$$

Then, for some integer $\lambda>\alpha+\frac{1}{2}$,

$$\left(\int_0^\pi\left|\sum_{k=0}^\infty g_k h_k R_k(\cos\theta)\left(\sin\frac{\theta}{2}\right)^{\gamma+\alpha+\frac{1}{2}}\left(\cos\frac{\theta}{2}\right)^{\beta+\frac{1}{2}}\right|^p d\theta\right)^{\frac{1}{p}} \leqq CI_\lambda.$$

Proof. Set

$$g(\theta)=\sum_{k=0}^\infty \Delta^{\lambda+1}g_k\sum_{j=0}^k A_{k-j}^\lambda h_j R_j(\cos\theta). \tag{3.5}$$

Then, by Szegő [8, Sec. 9.41], (3.5) and the substitution $x=\cos\theta$,

$$\|g\|_{p;\sigma,\tau}=\left(\int_0^\pi\left|g(\theta)\left(\sin\frac{\theta}{2}\right)^{\gamma+\alpha+\frac{1}{2}}\left(\cos\frac{\theta}{2}\right)^{\beta+\frac{1}{2}}\right|^p d\theta\right)^{1/p}\leqq$$

$$\leqq C\sum_{k=0}^\infty|\Delta^{\gamma+1}g_k|\sum_{j=0}^k j^{\alpha+\lambda+2}G_j(k,\lambda)\Big(\int_{-1}^1|P_j^{(\alpha+\lambda+1,\beta)}(x)(1-x)^{\left(\frac{\gamma}{2}+\frac{\alpha}{2}+\frac{1}{4}-\frac{1}{2p}\right)}\cdot$$

$$\cdot(1+x)^{\left(\frac{\beta}{2}+\frac{1}{4}-\frac{1}{2p}\right)}|^p\,dx\Big)^{1/p}$$

where $G_j(k,\lambda)$ is defined as in [8, (9.4.6)]. The above integral can be estimated with the aid of [8; Ex. 91] by $O(j^{\lambda-\gamma+1/2-1/p})$ and so, by [8; (9.41.7)],

$$\|g\|_{p;\sigma,\tau}\leqq C\sum_{k=0}^\infty|\Delta^{\lambda+1}g_k|\,k^{\lambda+\alpha+\frac{3}{2}-\gamma-\frac{1}{p}}.$$

Finally a standard argument shows that $g^\wedge(k)=g_k$.

Next we have to use Lemma 1 to estimate the $L^p_{(\sigma,\tau)}$ norm of certain functions $\Phi_n(\theta)$ arising from a partition of the unit sequence $\{1,1,\ldots\}$. Consider $\varphi_0\in C^\infty$ with compact support such that $0\leqq\varphi_0(t)\leqq1$,

$$\varphi_0(t)=\begin{cases}1, & 2^{-1/3}\leqq t\leqq 2^{1/3},\\ 0, & t\leqq 2^{-2/3}\ \text{ or }\ t\geqq 2^{2/3},\end{cases}$$

and $\sum_{n=0}^\infty\varphi_n(t)=1$ for $t\geqq1$ where we set $\varphi_n(t)=\varphi_0(2^{-n}t)$. Now define

$$\Phi_n(\theta)=\sum_{k=0}^\infty\varphi_n(k)h_kR_k(\cos\theta).$$

Then, for integer $\lambda>\alpha+\frac{1}{2}$,

$$\sum_{k=0}^\infty k^{\lambda+\alpha+\frac{3}{2}-\gamma-\frac{1}{p}}|\Delta^{\lambda+1}\varphi_n(k)|=O\left(2^{n\left(\alpha+\frac{3}{2}-\gamma-\frac{1}{p}\right)}\right)$$

and so, by Lemma 1,

$$\left(\int_0^\pi \left|\Phi_n(\theta)\left(\sin\frac{\theta}{2}\right)^{\gamma+\alpha+\frac{1}{2}}\left(\cos\frac{\theta}{2}\right)^{\beta+\frac{1}{2}}\right|^p d\theta\right)^{\frac{1}{p}} = O\left(2^{n\left(\alpha+\frac{3}{2}-\gamma-\frac{1}{p}\right)}\right). \tag{3.6}$$

Theorem 3. *Let* $1\leqq p\leqq 2$, $0<\gamma<\alpha+\frac{3}{2}-\frac{1}{p}$, $\alpha>-\frac{1}{2}$ *and* $\alpha\geqq\beta\geqq-\frac{1}{2}$. *If* $\sigma=\gamma+(2\alpha+1)\left(\frac{1}{2}-\frac{1}{p}\right)$ *and* $\tau=(2\beta+1)\left(\frac{1}{2}-\frac{1}{p}\right)$ *then* $M_p^p(\sigma,\tau;\alpha,\beta)\subset wbv_{p',\gamma}$.

Proof. Let $m\in M_p^p(\sigma,\tau;\alpha,\beta)$. For any $j\in N$ Minkowski's inequality and (3.6) give

$$\left(\sum_{2^{j-1}}^{2^j-1}\left|\sqrt{h_k}\Delta^\gamma m_k\right|^{p'}\right)^{1/p'} \leqq \sum_{n=j-1}^{\infty}\left(\sum_{2^{j-1}}^{2^j-1}\left|\sqrt{h_k}\Delta^\gamma(m_k\varphi_n(k))\right|^{p'}\right)^{1/p'} \leqq$$

$$\leqq C\|m\|_{M_p^p(\sigma,\tau;\alpha,\beta)}2^{j\left(\alpha+\frac{3}{2}-\gamma-\frac{1}{p}\right)}+C\sum_{n=j+2}^{\infty}\left(\sum_{2^{j-1}}^{2^j-1}\left|\sqrt{h_k}\Delta^\gamma(m_k\varphi_n(k))\right|^{p'}\right)^{1/p'}$$

and therefore

$$\left(\sum_{2^{j-1}}^{2^j-1}k^{-1}|k^\gamma\Delta^\gamma m_k|^{p'}\right)^{1/p'} \leqq \tag{3.7}$$

$$\leqq C\|m\|_{M_p^p(\sigma,\tau;\alpha,\beta)}+C2^{j\left(\gamma+\frac{1}{p}-\alpha-\frac{3}{2}\right)}\sum_{n=j+2}^{\infty}\left(\sum_{2^{j-1}}^{2^j-1}\left|\sqrt{h_k}\Delta^\gamma(m_k\varphi_n(k))\right|^{p'}\right)^{1/p'}.$$

But, since

$$\left|\Delta^\gamma(m_k\varphi_n(k))\right| = \left|\sum_{2^{n-2/3}\leqq l\leqq 2^{n+2/3}}A_{l-k}^{-\gamma-1}m_l\varphi_n(l)\right| = O(\|m\|_\infty 2^{-\gamma n})$$

the last term in (3.7) can be estimated by

$$C2^{j\left(\gamma+\frac{1}{p}-\alpha-\frac{3}{2}\right)}\|m\|_\infty 2^{-\gamma j}\left(\sum_{2^{j-1}}^{2^j-1}h_k^{p'/2}\right)^{1/p'} \leqq C\|m\|_\infty.$$

Noting that $\|m\|_\infty\leqq C\|m\|_{M_p^p(\sigma,\tau;\alpha,\beta)}$ we finally obtain

$$\left(\sum_{2^{j-1}}^{2^j-1}k^{-1}|k^\gamma\Delta^\gamma m_k|^{p'}\right)^{1/p'} \leqq C\|m\|_{M_p^p(\sigma,\tau;\alpha,\beta)}$$

uniformly in j, i.e. the assertion of Theorem 3.

Remarks. 1. In the unweighted case $\sigma=\tau=0$, we conclude that if $\alpha>-\frac{1}{2}$ and $1\leqq p<2$ then

$$M_p^p\left(0,0;\alpha,-\frac{1}{2}\right)\subset wbv_{p',\gamma},\quad \gamma=(2\alpha+1)\left(\frac{1}{p}-\frac{1}{2}\right). \tag{3.8}$$

2. Application of (3.8) to

$$m_k^{(n)}=\begin{cases} A_{n-k}^{\delta}/A_n^{\delta}, & 0\leqq k\leqq n,\\ 0, & k>n\end{cases}$$

and duality shows that in order for $\|m^{(n)}\|_{M_p^p\left(0,0;\alpha,-\frac{1}{2}\right)}$, $1<p<\infty$, to be uniformly bounded it is necessary that $\delta>(2\alpha+2)\left|\frac{1}{2}-\frac{1}{p}\right|-\frac{1}{2}$; this is only non-trivial as long as $\delta\geqq 0$ (which is a consequence of the trivial necessary condition: $l^\infty\supset M_p^p$). Note that $\delta>(2\alpha+2)\left|\frac{1}{2}-\frac{1}{p}\right|-\frac{1}{2}$ is sufficient [4, Sec. 8] for $\|m^{(n)}\|_{M_p^p(0,0;\alpha,\beta)}=O(1)$, $1\leqq p\leqq\infty$; in the case $p=1$, (3.8) leads only to the restriction $\delta\geqq\alpha+\frac{1}{2}$.

3. In [6] it is proved for $1<p<\infty$ that

$$wbv_{2,\lambda}\subset M_p^p(0,0;\alpha,\beta),\quad \lambda>\max\left\{(2\alpha+2)\left|\frac{1}{p}-\frac{1}{2}\right|,\ \frac{1}{2}\right\}. \tag{3.9}$$

Thus, when $\beta=-\frac{1}{2}$ and $1<p<2$ the required smoothness parameter λ in the sufficient condition (3.9) differs from the necessary smoothness parameter γ in (3.8) by any positive number larger than $\left(\frac{1}{p}-\frac{1}{2}\right)$. But the exact difference $\left(\frac{1}{p}-\frac{1}{2}\right)$ is needed for the embedding, i.e. $wbv_{2,\lambda}\subset wbv_{p',\mu}$ holds if $\lambda-\mu\geqq\left(\frac{1}{p}-\frac{1}{2}\right)$, $1<p\leqq 2$; see [4; Theorem 5].

4. Application of Askey's [1] transplantation theorem to Theorem 3 yields

$$M_p^p(0,0;\alpha,\beta)\subset wbv_{p',\gamma},\quad 1<p<2,$$

provided that $0<\gamma=(2\alpha+1)\left(\frac{1}{p}-\frac{1}{2}\right)<1-\frac{1}{p}$ and $0\leqq(2\beta+1)\left(\frac{1}{p}-\frac{1}{2}\right)<1-\frac{1}{p}$.

5. A little modification of Theorem 3 allows us to give necessary conditions for M_p^q multipliers, $1\leqq p<q\leqq 2$; for, arguing as in the proof of Theorem 3 it follows that

$$\left(\sum_{2^{j-1}}^{2^j-1}\left|\sqrt{h_k}\,\Delta^\gamma m_k\right|^{q'}\right)^{1/q'}\leqq C2^{j\left(\alpha+\frac{3}{2}-\gamma-\frac{1}{p}\right)}\|m\|_{M_p^q(\sigma,\tau;\alpha,\beta)}$$

which implies that

$$\left(\sum_{2^{j-1}}^{2^j-1}k^{-q'/p'}\left|k^\gamma\Delta^\gamma m_k\right|^{q'}\right)^{1/q'}\leqq C\|m\|_{M_p^q(\sigma,\tau;\alpha,\beta)};$$

that is, in the $wbv_{q,\gamma,\delta}$ notation of [5]

$$M_p^q(\sigma, \tau; \alpha, \beta) \subset wbv_{q',\gamma,1/q'-1/p'}, \quad 1 \leqq p < q \leqq 2,$$

when $\alpha \geqq \beta \geqq -\frac{1}{2}$, $\alpha > -\frac{1}{2}$, $0 < \gamma = \sigma + (2\alpha+1)\left(\frac{1}{q} - \frac{1}{2}\right)$, and $\tau = (2\beta+1)\left(\frac{1}{2} - \frac{1}{q}\right)$.

In particular,

$$M_p^q\left(0, 0; \alpha, -\frac{1}{2}\right) \subset wbv_{q',\gamma,1/q'-1/p'}, \quad 1 \leqq p < q \leqq 2,$$

when $0 < \gamma = (2\alpha+1)\left(\frac{1}{q} - \frac{1}{2}\right)$.

6. By analogous techniques one can also obtain necessary Hankel multiplier conditions.

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On the lattice of quasivarieties of distributive lattices with pseudocomplementation

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1. Introduction. In G. GRÄTZER and H. LAKSER [6] it was shown that not every quasivariety (implicational class) of distributive lattices with pseudocomplementation is a variety (equational class). In G. GRÄTZER [5] it was conjectured that there are $2^{\aleph_0}$ quasivarieties of distributive lattices with pseudocomplementation. This was proved to be the case by M. E. ADAMS [1] and, independently, by A. WROŃSKI [10].

In [4], V. A. GORBUNOV asked (Question 6) whether the lattice of quasivarieties of distributive lattices with pseudocomplementation is distributive.

K. B. LEE [7] showed that the lattice of varieties of distributive lattices with pseudocomplementation is a countable chain $\mathbf{B}_{-1}\subseteqq\mathbf{B}_0\subseteqq\mathbf{B}_1\subseteqq\ldots\subseteqq\mathbf{B}_n\subseteqq\ldots\subseteqq\mathbf{B}_\omega$. If B_n denotes the n-atom finite Boolean lattice and $\bar{B}_n$ is B_n with a new unit element, then $\bar{B}_n$, regarded as a distributive lattice with pseudocomplementation, generates the variety $\mathbf{B}_n$, $n\geqq 1$. $\mathbf{B}_0$ is the variety of Boolean algebras and $\mathbf{B}_{-1}$ is the trivial variety.

In this paper we sharpen the result of Adams and Wroński by showing that there are $2^{\aleph_0}$ quasivarieties in the variety $\mathbf{B}_3$. We also show that the lattice of quasivarieties in $\mathbf{B}_3$ is nonmodular, thereby answering Gorbunov's question.

Since the distributive lattices with pseudocomplementation $\bar{B}_0, \bar{B}_1$, and $\bar{B}_2$ are projective, all quasivarieties in $\mathbf{B}_2$ are varieties. Our results are thus optimal.

2. The Priestley duality. H. A. PRIESTLEY [8] established a duality between distributive lattices and certain partially-ordered (Hausdorff) topological spaces. M. E. ADAMS [1] investigated this duality for distributive lattices with pseudocomplementation. In this paper we need only consider finite distributive lattices with pseudocomplementation and so we can dispense with any considerations of topology.

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Let **L** denote the category of finite distributive lattices with pseudocomplementation; the morphisms preserve $0, 1$, and the operation of pseudocomplementation *. We denote by **P** the category whose objects are the finite posets and whose morphisms are isotone maps with the following property.

(*) If P_0, P_1 are posets, then $f: P_0 \to P_1$ is isotone and, for each $x \in P_0$ and $y \in P_1$, if $y \leqq f(x)$ and y is minimal in P_1, then there is a $z \in P_0$ with $z \leqq x$ and $f(z) = y$.

We call maps satisfying (*) *admissible maps*. There are a pair of contravariant functors $L: \mathbf{P} \to \mathbf{L}$ and $S: \mathbf{L} \to \mathbf{P}$. With each distributive lattice with pseudocomplementation L we associate the poset $S(L)$ of join-irreducible elements of L and for each homomorphism $f: L_0 \to L_1$ we define $S(f): S(L_1) \to S(L_0)$ by setting

$$S(f)(x) = \sup \{y \in L_0 | f(y) \leqq x\}.$$

With each finite poset P we associate the distributive lattice with pseudocomplementation $L(P)$ consisting of all hereditary subsets (including the empty set) of P. If $f: P_0 \to P_1$ is admissible we define $L(f): L(P_1) \to L(P_0)$ by setting $L(f)(x) = f^{-1}(x)$, where x is a hereditary subset of P_1.

Lemma 1 (M. E. Adams [1]). *$L: \mathbf{P} \to \mathbf{L}$ and $S: \mathbf{L} \to \mathbf{P}$ are contravariant functors, $HL: \mathbf{P} \to \mathbf{P}$ is naturally isomorphic to the identity functor and $LH: \mathbf{L} \to \mathbf{L}$ is naturally isomorphic to the identity functor.*

Lemma 2 (H. A. Priestley [8]). *Let P_0 and P_1 be finite posets and let $f: P_0 \to P_1$ be admissible. Then $L(f)$ is one-one if and only if f is surjective and $L(f)$ is surjective if and only if f is an embedding.*

Given two finite posets P_0, P_1 we denote their disjoint union (coproduct in **P**) by $P_0 \dot{\cup} P_1$. The following lemma is immediate.

Lemma 3. *If P_0, P_1 are finite posets then $L(P_0 \dot{\cup} P_1)$ is naturally isomorphic to $L(P_0) \times L(P_1)$.* If *$f_i: P_i \to P_0 \dot{\cup} P_1$, $i = 0$ or 1, is the natural embedding, then $L(f_i)$ corresponds to the projection $L(P_0) \times L(P_1) \to L(P_i)$.*

3. Steiner quasigroups. We shall consider posets that derive from partial Steiner triple systems. We proceed in an algebraic manner. A *partial Steiner quasigroup* is a partial algebra A endowed with a binary partial operation $\cdot$ satisfying the following three conditions.

i) For all $x \in A$, x^2 is defined and $x^2 = x$;

ii) for $x, y \in A$, xy is defined if and only if yx is defined, and in this case $xy = yx$;

iii) if $x, y \in A$ and xy is defined then so is $x(xy)$ and $x(xy) = y$.

We say that A is a *Steiner quasigroup* if the binary operation is defined everywhere.

With each partial Steiner quasigroup is associated, in an obvious way, a partial Steiner triple system — see, e.g., the Introduction to [9].

A mapping $f\colon A \to B$ of partial Steiner quasigroups is said to be a *homomorphism* if, whenever $x, y \in A$ and xy is defined, then $f(x) \cdot f(y)$ is defined and $f(x) \cdot f(y) = f(xy)$.

With each partial Steiner quasigroup A we associate a poset $P(A)$ of height 1. The elements of $P(A)$ are all subsets of A of the form $\{x, y, xy\}$ (i.e., all singletons and certain triples, namely, the blocks of the corresponding triple system), and the partial order is set containment, $\subseteq$. If $f\colon A \to B$ is a homomorphism we define $P(f)\colon P(A) \to P(B)$ by setting

$$P(f)(\{x, y, z\}) = \{f(x), f(y), f(z)\}.$$

Lemma 4. *If A and B are finite partial Steiner quasigroups and $f\colon A \to B$ is a homomorphism, then $P(f)\colon P(A) \to P(B)$ is an admissible map.*

Proof. $P(f)$ clearly maps $P(A)$ to $P(B)$ and it is equally clearly isotone. The minimal elements of $P(B)$ are the singletons $\{b\}$, $b \in B$. Let $\{x, y, z\} \in P(A)$ and let $b \in B$ with

$$\{b\} \subseteq P(f)(\{x, y, z\}) = \{f(x), f(y), f(z)\},$$

that is, with $b = f(x)$, say. Then $\{x\} \subseteq \{x, y, z\}$ and $P(f)(\{x\}) = \{b\}$; thus $P(f)$ is admissible, concluding the proof.

As a converse to Lemma 4 we have the following lemma.

Lemma 5. *Let A and B be finite partial Steiner quasigroups and let $g\colon P(A) \to P(B)$ be an admissible map. Then there is a unique homomorphism $f\colon A \to B$ with $g = P(f)$.*

Proof. Since g is admissible it maps minimal elements of $P(A)$ to minimal elements of $P(B)$. If $x \in A$, define $f(x) \in B$ by setting $\{f(x)\} = g(\{x\})$. The uniqueness of f is immediate, and we need only show that f is a homomorphism.

Let $x_0, x_1 \in A$ with $x_0 x_1$ defined; set $x_2 = x_0 x_1$. For each $i = 0, 1, 2$, $g(\{x_i\}) \subseteq g(\{x_0, x_1, x_2\})$ by the isotonicity of g. Thus $\{f(x_0), f(x_1), f(x_2)\} \subseteq g(\{x_0, x_1, x_2\})$. Let $y \in g(\{x_0, x_1, x_2\})$; by the admissibility of g there is an $i = 0, 1$, or 2 with $f(x_i) = y$. Consequently, $g(\{x_0, x_1, x_2\}) = \{f(x_0), f(x_1), f(x_2)\}$, and so $f(x_2) = f(x_0) \cdot f(x_1)$, showing that f is a homomorphism and concluding the proof.

4. The theorems. In the proof of our theorems we consider distributive lattices with pseudocomplementation of the form $LP(A)$, where A is a finite partial

Steiner quasigroup. We first show that such distributive lattices with pseudocomplementation reside in the correct variety.

Lemma 6. *If A is a finite partial Steiner quasigroup, then the distributive lattice with pseudocomplementation $LP(A)$ is a member of* $\mathbf{B}_3$.

Proof. Let T be the Steiner quasigroup consisting of three elements u, v, w with $uv=w$. Let $I=\{\langle a, b\rangle \in A^2 | ab$ is defined$\}$. For each $\langle a, b\rangle \in I$, define the homomorphism $f_{a,b}\colon T \to A$ by setting $f_{a,b}(u)=a, f_{a,b}(v)=b, f_{a,b}(w)=ab$. We then get a homomorphism $f\colon \dot{\bigcup}(T|I) \to A$, where $\dot{\bigcup}(T|I)$ denotes the partial Steiner quasigroup consisting of $|I|$ copies of T indexed by I, with xy defined if and only if x and y lie in the same copy of T. The resulting admissible map $P(f)$: $P(\dot{\bigcup}(T|I)) \to P(A)$ is surjective; if $\{a, b, c\} \in P(A)$, then $\{a, b, c\}=P(f_{a,b})(\{u, v, w\})$. Since $P(\dot{\bigcup}(T|I))=\dot{\bigcup}(P(T)|I)$ we see, by Lemma 2 and 3, that $LP(A)$ is a subalgebra of $(LP(T))^I$. But $LP(T)\cong \bar{B}_3$; thus $LP(A) \in \mathbf{B}_3$, proving the lemma.

We now recall the characterization of the quasivariety generated by a class of algebras.

Lemma 7 (G. Grätzer and H. Lakser [6]). *Let* $\mathbf{K}$ *be a class of algebras. An algebra A is a member of the quasivariety generated by* $\mathbf{K}$ *if and only if A is isomorphic to a subalgebra of a product of ultraproducts of families of algebras in* $\mathbf{K}$.

As immediate corollaries we get the following two lemmas.

Lemma 8. *Let $(\mathbf{K}_i | i \in I)$ be a family of quasivarieties of distributive lattices with pseudocomplementation, and let P be a finite poset. The distributive lattice with pseudocomplementation $L(P)$ is a member of the quasivariety* $\mathbf{K}$ *generated by the family $(\mathbf{K}_i | i \in I)$ if and only if there are a finite subset $I_0 \subseteq I$, finite posets P_i, $i \in I_0$, with $L(P_i) \in \mathbf{K}_i$, and an admissible surjection $f\colon \dot{\bigcup}(P_i | i \in I_0) \to P$.*

Proof. Since $L(P)$ is finite, Lemma 7 implies that $L(P) \in \mathbf{K}$ if and only if there are a finite subset $I_0 \subseteq I$ and finite distributive lattices with pseudocomplementation $L_i \in \mathbf{K}_i$, $i \in I_0$, such that $L(P)$ is a subalgebra of $\Pi(L_i | i \in I_0)$. The lemma is then immediate from Lemmas 1, 2 and 3.

Lemma 9. *Let $(P_i | i \in I)$ be a family of finite posets and let P be a finite poset. The distributive lattice with pseudocomplementation $L(P)$ is a member of the quasivariety generated by $(L(P_i) | i \in I)$ if and only if there is a finite subset $I_0 \subseteq I$ and an admissible surjection $f\colon \dot{\bigcup}(P_i | i \in I_0) \to P$.*

A Steiner quasigroup is said to be *planar* if it has at least four elements and any three distinct elements a, b, c with $ab \neq c$ generate the whole quasigroup. J. Doyen [3] showed that for each $n \geqq 7$ with $n \equiv 1$ or $3 \pmod 6$ there is a planar Steiner quasigroup of cardinality n. R. W. Quackenbush [9] proved that a finite planar Steiner quasigroup is simple if its cardinality is not 9. Since any four distinct elements of a planar Steiner quasigroup generates the quasigroup, we immediately get the following lemma.

Lemma 10. *Let* A *and* B *be nonisomorphic finite planar Steiner quasigroups with* $|A|\neq 9$, *and let* $f\colon A\to B$ *be a homomorphism. Then* f *is trivial, that is,* $\operatorname{Im} f$ *is a singleton.*

Theorem 1. *There are* $2^{\aleph_0}$ *quasivarieties of distributive lattices with pseudocomplementation in* $\mathbf{B}_3$.

Proof. Let $I=\{n\geqq 7 \mid n\equiv 1 \text{ or } 3 \pmod 6 \text{ and } n\neq 9\}$, and, for each $n\in I$, let A_n be a Steiner quasigroup of cardinality n. For each subset $J\subseteq I$ let $\mathbf{Q}(J)$ be the quasivariety of distributive lattices with pseudocomplementation generated by $(LP(A_n) \mid n\in J)$. By Lemma 6, $\mathbf{Q}(J)\subseteq \underset{\sim}{B}_3$. We claim that if $J_0\neq J_1$ then $\mathbf{Q}(J_0)\neq \neq\mathbf{Q}(J_1)$. Indeed, let $i\in J_1-J_0$ and assume that $LP(A_i)\in \mathbf{Q}(J_0)$. Let $a_0, a_1, a_2\in A_i$ with $a_0\neq a_1$ and $a_0a_1=a_2$. By Lemma 9, there is an $n\in J_0$ and an admissible map $g\colon P(A_n)\to P(A_i)$ with $\{a_0, a_1, a_2\}\in \operatorname{Im} g$. By Lemma 5, there is a homomorphism $f\colon A_n\to A_i$ with $g=P(f)$. Now, f is not trivial since $\{a_0, a_1, a_2\}\subseteq \operatorname{Im} f$, contradicting Lemma 10.

Thus the quasivarieties $\mathbf{Q}(J)$ are distinct for distinct $J\subseteq I$, proving the theorem.

Theorem 2. *The lattice of quasivarieties of distributive lattices with pseudocomplementation in* $\mathbf{B}_3$ *is not modular.*

Proof. A. I. Budkin and V. A. Gorbunov [2] proved that the lattice of quasivarieties of algebras in a variety is modular if and only if it is distributive; thus we need only establish nondistributivity.

Let A_0, A_1 be nonisomorphic finite planar Steiner quasigroups, neither of cardinality 9; each is thus simple. For $i=0, 1$, let $a_i, b_i, c_i\in A_i$ be distinct with $c_i=a_ib_i$. Let B by the partial Steiner quasigroup obtained by amalgamating A_0 and A_1 over $\{a_i, b_i, c_i\}$. More specifically,

$$B=(A_0-\{a_0, b_0, c_0\})\,\dot{\cup}\,(A_1-\{a_1, b_1, c_1\})\,\dot{\cup}\,\{a, b, c\}$$

with a, b, c distinct and the operation defined as follows.

i) $ab=c$;

ii) if $x\in A_i-\{a_i, b_i, c_i\}$, $i=0$ or 1, then $xa=xa_i$ (in A_i), $xb=xb_i$, and $xc=xc_i$;

iii) if $x_0, x_1\in A_i-\{a_i, b_i, c_i\}$, $i=0$ or 1; then x_0x_1 is defined as in A_i;

iv) if $x_i\in A_i-\{a_i, b_i, c_i\}$, $i=0, 1$, then x_0x_1 is undefined.

Then B is a partial Steiner quasigroup and we have the obvious embeddings $A_i\to B$, $i=0, 1$; denote the images by A_i'. Thus $A_i'=(A_i-\{a_i, b_i, c_i\})\cup\{a, b, c\}$.

Claim 1. *Let* $i=0$ *or* 1 *and let* $f\colon A_i\to B$ *be a nontrivial homomorphism. Then* f *is one-one and* $\operatorname{Im} f=A'$.

By condition iv) every (total) subalgebra of B is a subalgebra of A_0' or A_1'. The claim then follows by Lemma 10, recalling that A_i is simple.

Now, for $i=0, 1$, let $\mathbf{A}_i$ be the quasivariety generated by $LP(A_i)$, and let $\mathbf{B}$ be the quasivariety generated by $LP(B)$.

Claim 2. $LP(B)\in\mathbf{A}_0\vee\mathbf{A}_1$ *(the join denoting the join in the lattice of quasi-varieties).*

The natural homomorphism of partial Steiner quasigroups $A_0\dot\cup A_1\to B$ yields an admissible map $P(A_0)\dot\cup P(A_1)\to P(B)$. This map is surjective by condition iv); consequently, $LP(B)$ is isomorphic to a subalgebra of $LP(A_0)\times LP(A_1)$, establishing the claim.

Claim 3. $LP(B)\notin(\mathbf{A}_0\wedge\mathbf{B})\vee(\mathbf{A}_1\wedge\mathbf{B})$.

Assume, to the contrary, that $LP(B)\in(\mathbf{A}_0\wedge\mathbf{B})\vee(\mathbf{A}_1\wedge\mathbf{B})$. By Lemma 8, there are finite posets P_0, P_1 with $L(P_i)\in\mathbf{A}_i\wedge\mathbf{B}$, $i=0, 1$, and an admissible surjection $P_0\dot\cup P_1\to P(B)$. Thus, for some $i=0$ or 1, there is an admissible map $f\colon P_i\to P(B)$ with $\{a, b, c\}\in\operatorname{Im} f$, that is, $\{a, b, c\}=f(u)$ for some $u\in P_i$. We fix this i for the rest of the argument.

Since $L(P_i)\in\mathbf{B}$ we conclude, by Lemma 9, that there is an element $\{e_0, e_1, e_2\}\in\in P(B)$ and an admissible map $g\colon P(B)\to P_i$ with $u=g(\{e_0, e_1, e_2\})$. Now there is a homomorphism $\varphi\colon B\to B$ with $P(\varphi)=fg$. Thus $\{\varphi e_0, \varphi e_1, \varphi e_2\}=\{a, b, c\}$ and so $\varphi e_0\neq\varphi e_1$. Since $e_0e_1\,(=e_2)$ is defined, both e_0 and e_1 are elements of A_j', $j=0$ or 1, and so φ is nontrivial on A_j'. By Claim 1, φ is one-one on A_j' and, since $\{a, b, c\}\subseteq A_0'\cap A_1'$, φ is also nontrivial on A_k', $k\neq j$. Thus, by Claim 1 again, $\varphi(A_0')=A_0'$ and $\varphi(A_1')=A_1'$. By condition iv) $fg=P(\varphi)\colon P(B)\to P(B)$ is surjective. Thus f is surjective.

Now let $x\in B-A_i'$; ax is defined and so there is a $v\in P_i$ with $f(v)=\{a, x, ax\}$. Since $L(P_i)$ is also a member of $\mathbf{A}_i$ there is, by Lemma 9, an admissible map $h\colon P(A_i)\to P_i$ with $v\in\operatorname{Im} h$. The admissible map $fh\colon P(A_i)\to P(B)$ is then of the form $P(\psi)$ for some homomorphism $\psi\colon A_i\to B$. Since $\{a, x, ax\}\subseteq\operatorname{Im}\psi$ $\psi\colon A_i\to B$ is nontrivial; since $x\notin A_i'$, we derive a contradiction to Claim 1. Thus our assumption that $LP(B)\in(\mathbf{A}_0\wedge\mathbf{B})\vee(\mathbf{A}_1\wedge\mathbf{B})$ is false, verifying Claim 3.

From Claim 2, $\mathbf{B}\leqq\mathbf{A}_0\vee\mathbf{A}_1$ and from Claim 3, $\mathbf{B}\not\leqq(\mathbf{A}_0\wedge\mathbf{B})\vee(\mathbf{A}_1\wedge\mathbf{B})$. Thus the lattice of quasivarieties in $\mathbf{B}_3$ is not distributive, concluding the proof of Theorem 2.

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On empirical Prékopa processes

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1. Introduction. In 1963 PRÉKOPA [8] considered the following inventory model. Given a time period T during which we observe the production of a factory A. In its production factory A uses a type of a material with constant intensity c. So in the given period T it needs the amount cT of this material. This should be supplied by factory B on the basis of the following contract.

a) For a fixed number λ, $0\leqq\lambda\leqq 1$, B will deliver the λ-portion λcT of the whole amount cT at n time-points $t_1, \ldots, t_n$, each time the amount $\lambda\frac{cT}{n}$. These instants t_i are independent random variables (r.v.'s) uniformly distributed on $(0, T)$.

b) The remaining portion $(1-\lambda)cT$ will be delivered at n time-points $r_1, \ldots, r_n$ which are again independent r.v.'s uniformly distributed on $(0, T)$, in amounts $s_1, \ldots, s_n$, respectively. These amounts are also r.v.'s, they are uniform spacings of the interval $(0, (1-\lambda)cT)$, and the sequence $(s_1, \ldots, s_n)$ is independent of both $(r_1, \ldots, r_n)$ and $(t_1, \ldots, t_n)$.

As usual, the spacing variables are constructed as follows. Divide the interval $(0, (1-\lambda)cT)$ into n subintervals by $n-1$ independent, uniformly distributed r.v.'s. Then $s_1, \ldots, s_n$ are the resulting lengths of these subintervals, and they will be referred to as "random additions".

Factory A wishes to avoid lack of material, so it needs an initial stock M_λ (at $t=0$) to balance with high prescribed probability the uncertainty in the delivery. In order to formulate exactly what M_λ is, we have to introduce the following quantities. Let $q_1, \ldots, q_{n-1}$ be independent r.v.'s uniformly distributed on $(0, cT)$, and let $0\leqq t_1^*\leqq\ldots\leqq t_n^*\leqq T$, $0\leqq r_1^*\leqq\ldots\leqq r_n^*\leqq T$ and $0\leqq q_1^*\leqq\ldots\leqq q_{n-1}^*\leqq cT$ denote the respective order statistics corresponding to the sequences t, r, and q. If we

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introduce the stochastic processes

$$R_n(t) = \begin{cases} 0, & \text{for } 0 \leqq t \leqq r_1^*, \\ q_k^*, & \text{for } r_k^* < t \leqq r_{k+1}^* \quad (k = 1, 2, \ldots, n-1), \\ cT, & \text{for } r_n^* < t \leqq T, \end{cases}$$

$$K_n(t; \lambda, c, T) = \begin{cases} (1-\lambda)R_n(t), & \text{for } 0 \leqq t \leqq t_1^*, \\ \lambda \dfrac{k}{n} cT + (1-\lambda) R_n(t), & \text{for } t_k^* < t \leqq t_{k+1}^* \quad (k = 1, 2, \ldots, n-1), \\ \lambda cT + (1-\lambda) R_n(t), & \text{for } t_n^* < t \leqq T, \end{cases}$$

then $(1-\lambda)R_n(t)$ and $K_n(t; \lambda, c, T)$ represent the amount of random additions and the total amount delivered up to time t, respectively. Let $\varepsilon>0$. Prékopa's problem is: what initial stock $M_\lambda = M(\varepsilon, \lambda, c, T, n)$ should A posses to ensure the continuous production with probability $1-\varepsilon$. In order to obtain a solution we therefore need to know the probability

$$p_n(\lambda) = P(\sup_{0 \leqq t \leqq T} (ct - K_n(t; \lambda, c, T)) < M_\lambda) =$$

$$= P\left(\sup_{0 \leqq t \leqq T} \left(\frac{t}{T} - \frac{1}{cT} K_n(t; \lambda, c, T)\right) < \frac{M_\lambda}{cT}\right)$$

to find at least an asymptotic solution M_λ of the reliability equation

$$p_n(\lambda) = 1-\varepsilon. \tag{1.1}$$

2. Summary. The form of the latter probability suggests to simplify the whole model. Following Prékopa [8], let

$$X = (X_1, X_2, \ldots), \quad Y = (Y_1, Y_2, \ldots), \quad Z = (Z_1, Z_2, \ldots)$$

be three sequences of independent r.v.'s uniformly distributed on $(0, 1)$. For fixed n, $X_1^* \leqq \ldots \leqq X_n^*$, $Y_1^* \leqq \ldots \leqq X_n^*$, $Z_1^* \leqq \ldots \leqq Z_n^*$ are the three corresponding ordered samples. Sequences X, Y, and Z will correspondingly play the role of the former sequences $(t_1, t_2, \ldots)$, $(r_1, r_2, \ldots)$, $(q_1, q_2, \ldots)$.

In the original model of Prékopa the delivery times of the fixed amounts and the random additions were identical $(t_1 = r_1, \ldots, t_n = r_n)$, and consequently in the simplified model he had $X=Y$. Csörgő [5] considered the possibility $X \neq Y$, assuming that X and Y are independent. The aim of the present paper is to study the general model when X and Y can depend on each other, i.e., to "bridge" the two extreme cases considered by Prékopa and later by Csörgő.

Denote by $F_n(t; X)$, $F_n(t; Y)$ and $F_n(t; Z)$ the n-th stage empirical distribution functions corresponding to the sequences X, Y and Z, respectively. If

$$\psi(x) = \begin{cases} 0, & \text{for } x < 0, \\ 1, & \text{for } x \geqq 0 \end{cases}$$

then, for instance, $F_n(t;X)=\frac{1}{n}\sum_{i=1}^{n}\psi(t-X_i)$. The following equivalent form for F_n will be used later. Clearly $\psi(x)=\frac{1}{2}\left(\frac{|x|}{x}+1\right)$, if $x\neq 0$. Since the distribution function of the X_k variables is continuous, for each fixed $t\in[0,1]$ we have almost surely that

$$F_n(t;X)=\frac{1}{2n}\sum_{i=1}^{n}\left(\frac{|t-X_i|}{t-X_i}+1\right).$$

Define the stochastic process

$$I_n(t)=\begin{cases}0, & \text{if } 0\leqq t\leqq X_1^*,\\ Z_k^*, & \text{if } X_k^*<t\leqq X_{k+1}^* \quad (k=1,2,\dots,n-1),\\ 1, & \text{if } X_n^*<t\leqq 1,\end{cases}$$

and for an arbitrarily fixed λ $(0\leqq\lambda\leqq 1)$ consider

$$X_n^{f(\lambda)}(t)=\big(n/f(\lambda)\big)^{1/2}\big(t-K_n(t;\lambda)\big)=\big(n/f(\lambda)\big)^{1/2}\big(t-\lambda F_n(t;Y)-(1-\lambda)I_n(t)\big),$$

where $f(\lambda)$ is an arbitrary function on the interval $[0,1]$ such that

$$\inf_{0\leqq\lambda\leqq 1}f(\lambda)=\lambda^*>0. \tag{2.1}$$

In his first paper Prékopa [8] made an assertion (if $X=Y$) concerning the limit distribution of $\sup\limits_{0\leqq t\leqq 1}X_n^{1+(1-\lambda)^2}(t)$, which reduced to Smirnov's classical result when $\lambda=1$. Later in [9] he proved more, namely that

$$X_n^{1+(1-\lambda)^2}(\cdot)\xrightarrow{\mathscr{D}}B(\cdot),\quad X=Y, \tag{2.2}$$

where $\xrightarrow{\mathscr{D}}$ denotes weak convergence in Skorohod's $D[0,1]$ space and $B(t)$, $0\leqq t\leqq 1$, is the Brownian bridge process (cf. Billingsley [1]). Csörgő [5] noticed that the $X_n^{f(\lambda)}(t)$ process admits the following more convenient representation:

$$\begin{aligned}X_n^{f(\lambda)}(t)&=\lambda\big(n/f(\lambda)\big)^{1/2}\big(t-F_n(t;Y)\big)+(1-\lambda)\big(n/f(\lambda)\big)^{1/2}\big(t-F_n(t;X)\big)+\\&\quad+(1-\lambda)\big(f(\lambda)\big)^{-1/2}q_n\big(F_n(t;X);Z^{-1}\big)\end{aligned} \tag{2.3}$$

where

$$q_n(t;Z^{-1})=\sqrt{n}\big(t-F_n(t;Z^{-1})\big)$$

is the uniform quantile process, i.e.,

$$F_n(t;Z^{-1})=\inf\big(x\in[0,1]\colon F_n(x;Z)\geqq t\big).$$

Using (2.3) he gave an easy proof of (2.2) and also proved that

$$X_n^{\lambda^2+2(1-\lambda)^2}(\cdot)\xrightarrow{\mathscr{D}}B(\cdot)\quad\text{with}\quad X,Y\ \text{independent}. \tag{2.4}$$

Assuming a general condition on the dependency structure of sequences X and Y, we prove in Section 3 a general weak convergence theorem. The limit process is Gaussian, but it is not always a Brownian bridge. A necessary and sufficient condition is given to ensure that the limit process be the Brownian bridge. So (2.2) and (2.4) become corollaries of the general theorem. In Section 4 we apply the general weak convergence result to answer the original question, i.e., to determine (asymptotically) the required initial stock M_λ for the continuous production. Following Prékopa we generalize our general model in Section 5 to the case when the consumption of the delivered material in factory A is the same type random process as the delivery process. In Section 6 we come back to the two special cases in (2.2), (2.4) and apply recent strong approximation results to approximate $X_n^{f(\lambda)}$ by a sequences of appropriate Brownian bridges. This result will provide information about the accuracy of the asymptotic solutions of our reliability equations.

3. Weak convergence of the process $X_{n(\cdot)}^{f(\lambda)}$. According to our assumption in the original (non-simplified) model, we assume throughout that the sequence Z is independent of both sequences X and Y. It is also assumed throughout that the two dimensional random vectors (X_1, Y_1), (X_2, Y_2)... are independent. Define the distribution function of the pair (X_i, Y_i) $(i=1, 2, ...)$

$$P(X_i < t,\ Y_i < s) = G_i(t, s).$$

Theorem 3.1. *Suppose*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (G_i(t, s)+G_i(s, t)) = G(t, s)$$

exists for every $t, s \in [0, 1]$. *Then*

$$X_n^{f(\lambda)}(\cdot) \xrightarrow{\mathcal{D}} X^{f(\lambda)}(\cdot)$$

where $X^{f(\lambda)}$ *is a Gaussian process on* $[0, 1]$ *with* $EX^{f(\lambda)}(t)=0$ *and*

$$EX^{f(\lambda)}(t)X^{f(\lambda)}(s) = (f(\lambda))^{-1}[\lambda(1-\lambda)(G(t, s)-2t)+t(1-s)+(1-\lambda)^2 t(1-s)], \quad t \leqq s.$$

Proof. Since Z is independent of X and Y, the limit process of the third term of (2.3) and that of the sum of the first two terms are independent. As it is well known, the uniform quantile process $q_n(t; Z^{-1})$ converges weakly to the Brownian bridge, therefore it is enough to study the limit behaviour of the process

$$\tilde{X}_n^{f(\lambda)}(t) = (n/f(\lambda))^{1/2}[\lambda(t-F_n(t; Y))+(1-\lambda)(t-F_n(t; X))].$$

First we prove that the finite dimensional distributions of $\tilde{X}_n^{f(\lambda)}$ converge to those of the Gaussian process $\tilde{X}^{f(\lambda)}$ with $E\tilde{X}^{f(\lambda)}(t)=0$ and

$$E\tilde{X}^{f(\lambda)}(t)\tilde{X}^{f(\lambda)}(s) = (f(\lambda))^{-1}[\lambda(1-\lambda)(G(t, s)-2t)+t(1-s)], \quad t \leqq s.$$

In order to do this we need the following easy equation

$$(3.1)\qquad P\big((t-X_i)(s-Y_i)>0\big)+P\big((s-X_i)(t-Y_i)>0\big)= \\ =2[G_i(t,s)+G_i(s,t)]+2-2(t+s),\quad t\leqq s,$$

and the identity

$$\tilde{X}_n^{f(\lambda)}(t)=(nf(\lambda))^{-1/2}\left[\sqrt{n}t-\sum_{i=1}^{n}\left(\frac{\lambda}{2}\left(\frac{|t-Y_i|}{t-Y_i}+1\right)+\frac{1-\lambda}{2}\left(\frac{|t-X_i|}{t-X_i}+1\right)\right)\right]$$

taking place with probability one. We need to know the following expectations computed by (3.1)

$$(3.2)\qquad E\frac{|t-X_i|}{t-X_i}=E\frac{|t-Y_i|}{t-Y_i}=2t-1,$$

$$E\frac{|t_j-X_i|}{t_j-X_i}\cdot\frac{|t_l-X_i|}{t_l-X_i}=E\frac{|t_j-Y_i|}{t_j-Y_i}\cdot\frac{|t_l-Y_i|}{t_l-Y_i}=2P\big((t_j-X_i)(t_l-X_i)>0\big)-1= \\ =2(t_j-t_l)+1,\quad t_j\leqq t_l,$$

$$E\frac{|t_j-Y_i|}{t_j-Y_i}\cdot\frac{|t_l-X_i|}{t_l-X_i}=2P\big((t_j-Y_i)(t_l-X_i)>0\big)-1.$$

Taking the time-points $t_1,\dots,t_k$ $(0\leqq t_1<t_2<\dots<t_k\leqq 1)$ and real numbers $a_1,\dots,a_k$ by the Cramér—Wold device (BILLINGSLEY [1], p. 49) we must show that

$$(3.3)\qquad \sum_{j=1}^{k}a_j\tilde{X}_n^{f(\lambda)}(t_j)\xrightarrow{\mathcal{D}}\sum_{j=1}^{k}a_j\tilde{X}^{f(\lambda)}(t_j).$$

Here $\xrightarrow{\mathcal{D}}$ stands, naturally, for convergence in distribution on the real line. The left hand side of (3.3) can be written as

$$(nf(\lambda))^{1/2}\sum_{j=1}^{k}a_j\left[\sqrt{n}t_j-\sum_{i=1}^{n}\left(\frac{\lambda}{2}\left(\frac{|t_j-Y_i|}{t_j-Y_i}+1\right)+\frac{1-\lambda}{2}\left(\frac{|t_j-X_i|}{t_j-X_i}+1\right)\right)\right]= \\ =(f(\lambda))^{-1/2}\sum_{j=1}^{k}a_jt_j-(nf(\lambda))^{-1/2}\sum_{i=1}^{n}\alpha_i,$$

where the r.v. α_i is defined by

$$\alpha_i=\sum_{j=1}^{k}a_j\left(\frac{\lambda}{2}\left(\frac{|t_j-Y_i|}{t_j-Y_i}+1\right)+\frac{1-\lambda}{2}\left(\frac{|t_j-X_i|}{t_j-X_i}+1\right)\right)=\sum_{j=1}^{k}a_j\gamma_{j,i}.$$

$\alpha_1,\alpha_2,\dots$ are independent because the random vectors $(X_1,Y_1),(X_2,Y_2),\dots$ are independent. Applying the equalities in (3.2), we get $E\alpha_i=\sum_{j=1}^{k}a_jt_j$ $(i=1,2,\dots)$. Also, if $t_j<t_l$, then by (3.1) and (3.2) we find for the products of the terms in α_i the following

$$E\gamma_{j,i}\gamma_{l,i}=\lambda^2t_j+(1-\lambda)^2t_j+\lambda(1-\lambda)[G_i(t_j,t_l)+G_i(t_l,t_j)].$$

Consequently,

$$D^2\Big((nf(\lambda))^{-1/2}\sum_{i=1}^{k}\alpha_i\Big)=(nf(\lambda))^{-1}\sum_{i=1}^{n}D^2\Big(\sum_{j=1}^{k}a_j\gamma_{j,i}\Big)=$$

$$=(nf(\lambda))^{-1}\sum_{i=1}^{n}\Big(\sum_{j=1}^{k}a_j^2D^2\gamma_{j,i}+2\sum_{l<j}a_la_j(E\gamma_{l,i}\gamma_{j,i}-E\gamma_{l,i}E\gamma_{j,i})\Big)=$$

$$=(f(\lambda))^{-1}\sum_{j=1}^{k}a_j^2\Big(2\lambda(1-\lambda)\Big(n^{-1}\sum_{i=1}^{n}G_i(t_j,t_j)-t_j\Big)+t_j-t_j^2\Big)+$$

$$+2(f(\lambda))^{-1}\sum_{l<j}a_la_j\Big(\frac{\lambda(1-\lambda)}{n}\sum_{i=1}^{n}(G_i(t_l,t_j)+G_i(t_j,t_l))+t_j(1-t_l)-2\lambda(1-\lambda)t_j\Big).$$

We saw that $\alpha_1,\alpha_2,\ldots$ are independent, and it is easy to see that the moments $E|\alpha_i-E\alpha_i|^3$ are bounded. Hence the central limit theorem ([10], p. 442) can be used to finish the proof of (3.3).

Now we show that the sequence $X_n^{f(\lambda)}$ is tight. Since $X_n^{f(\lambda)}(0)=0$, it is enough to prove (cf. Theorem 15.5 of Billingsley [1]) that for each positive ε we have

$$\lim_{c\to 0}\overline{\lim_{n\to\infty}}P\big(\sup_{|s-t|\le c}|X_n^{f(\lambda)}(s)-X_n^{f(\lambda)}(t)|>\varepsilon\big)=0. \tag{3.4}$$

Besides the already introduced quantile process, let us also introduce the empirical processes $\beta_n(t;U)=\sqrt{n}\,(F_n(t;U)-t)$, $U=X,Y$. We have

$$P\big(\sup_{|s-t|\le c}|X_n^{f(\lambda)}(s)-X_n^{f(\lambda)}(t)|>\varepsilon\big)\le P\Big(\sup_{|s-t|\le c}|\beta_n(s;Y)-\beta_n(t;Y)|>\frac{\varepsilon}{2}\sqrt{\lambda^*}\Big)+$$

$$+P\Big(\sup_{|s-t|\le c}|\beta_n(s;X)-\beta_n(t;X)|>\frac{\varepsilon}{4}\sqrt{\lambda^*}\Big)$$

$$+P\Big(\sup_{|s-t|\le 2c}|q_n(s;Z^{-1})-q_n(t;Z^{-1})|>\frac{\varepsilon}{4}\sqrt{\lambda^*}\Big)+$$

$$+P\big(\sup_{0\le u\le 1}|F_n(u;X)-u|>c\big),$$

where λ^* is of (2.1). Using the Glivenko—Cantelli theorem and the fact that the empirical and quantile processes satisfy condition (3.4), the tightness of $X_n^{f(\lambda)}$ is clear.

Having the form of the covariance function given in Theorem 3.1, one obtains the following

Corollary 3.2. *Under the conditions of Theorem 3.1, the process* $X_n^{f(\lambda)}$ *converges weakly to the Brownian bridge for every* λ, $0\le\lambda\le 1$ *if and only if the following two conditions are satisfied*

(i) $$G(t,s)=kt(1-s)+2t,\quad t\le s$$

where k *is a fixed real number with* $-2 \leqq k \leqq 0$

(ii) $$f(\lambda) = 1 + k\lambda(1-\lambda) + (1-\lambda)^2.$$

The following simple example shows that one can indeed have a limit process in Theorem 3.1 which is not a Brownian bridge. If $G_i(t, s) = t - t^2(1-s)$ $(t \leqq s)$, then

$$EX^{f(\lambda)}(t)X^{f(\lambda)}(s) = t(1-s)\big(1+(1-\lambda)^2 - 2\lambda(1-\lambda)t\big)\big(f(\lambda)\big)^{-1}, \quad t \leqq s,$$

and clearly there is no such $f(\lambda)$, for which the latter would be the covariance function of the Brownian bridge.

If the variables X_i, Y_i are identical then $k=0$, if the variables X_i, Y_i are independent then $k=-2$, thus (2.2) of Prékopa [9] and (2.4) of Csörgő [5] follow from Theorem 3.1. Also, it follows from Theorem 3.1 that the processes $X_n^{f(1)}$ and $X_n^{f(0)}$ converge to a Brownian bridge, if $f(1)=1$ and $f(0)=2$ for every function $G(t, s)$. Indeed, in the first case our process is merely the classical empirical process (on the Y sequences), while in the second case "the empirical process with random jumps". It can also be noted here that László [7] managed to compute the exact distribution of the supremum of the process $X_n^{f(0)}$, $f(0)=2$.

4. Application to the solution of the reliability equation. The result of the preceding section can be applied to obtain an asymptotic solution of the reliability equation (1.1).

Theorem 4.1. *If the conditions of Corollary* 3.2 *are true, then the asymptotic solution of the reliability equation is*

$$M_\lambda \approx cT\left(\frac{f(\lambda)}{2n}\log\frac{1}{\varepsilon}\right)^{1/2}.$$

Proof. The reliability equation can be written in the form

$$P\left(\sup_{0\leqq t\leqq 1}\left(\frac{n}{f(\lambda)}\right)^{1/2}(t-K_n(t;\lambda)) < \frac{M_\lambda}{cT}\left(\frac{n}{f(\lambda)}\right)^{1/2}\right) = 1-\varepsilon.$$

By Corollary 3.2 $P\left(\sup\limits_{0\leqq t\leqq 1}\left(\frac{n}{f(\lambda)}\right)^{1/2}(t-K_n(t;\lambda))<x\right)$ converges to

$$P\big(\sup_{0\leqq t\leqq 1} B(t) < x\big) = \begin{cases} 0, & \text{if } x \leqq 0 \\ 1-\exp(-2x^2), & \text{if } x > 0, \end{cases}$$

Therefore, for large enough n, the reliability equation is $1-\exp\left(-2\left(\frac{M_\lambda}{cT}\right)^2\frac{n}{f(\lambda)}\right)=$ $=1-\varepsilon$, and hence the theorem.

Let us imagine that the "dependency constant" k, $-2\leqq k\leqq 0$, (between the transportation instants of the fixed amounts and the random additions) has already been determined (probably by some independent statistical procedure). The minimum of $f(\lambda)=f_k(\lambda)$ is attained at $\lambda_k=(2-k)/2(1-k)$, and $f_k(\lambda_k)=$ $=(k+5-1/(1-k))/4$. Regarding $f_k(\lambda_k)$ as a function of k we see that it strictly increases on $[-2, 0]$. Consequently, the initial stock is minimal if the random variables X_i, Y_i are independent, and it is maximal if the r.v.'s X_i, Y_i are identical. At first it may seem surprising that the initial stock is minimal, when the delivery process is "most unorganized". On the other hand this is intuitively clear if we think that there are $2n$ independent deliveries in this case. Now we also have a concrete measure of this intuitive feeling. Since the maximum is $M_{\lambda_0}=cT\left(\log\frac{1}{\varepsilon}\Big/2n\right)^{1/2}$ and the minimum is $M_{\lambda_{-2}}=cT\left(\log\frac{1}{\varepsilon}\Big/3n\right)^{1/2}$, the proportion of the minimal and the maximal stock is $(2/3)^{1/2}\approx 0{,}82$.

5. Random consumption. We interpret the consumption process $T_n(t;\mu)$ similarly as we interpreted the arrival process $K_n(t;\lambda)$ in Section 1. The function $H_i(t, s)$ and $H(t, s)$ are defined as we defined the functions $G_i(t, s)$ and $G(t, s)$ there, respectively. We assume that the process $T_n(t;\mu)$ and $K_n(t;\lambda)$ are independent for every n. In this case the reliability equation is

$$P\left(\sup\,(T_n(t;\mu)-K_n(t;\lambda))<\frac{M_{\lambda,\mu}}{K}\right)=1-\varepsilon \tag{5.1}$$

where K is the total amount of the material used by the factory in its production.

Theorem 5.1. *The process* $(n/(f(\lambda)+g(\mu)))^{1/2}(T_n(t;\mu)-K_n(t;\lambda))$ *converges weakly to the Gaussian process* $Z(t)$ *with* $EZ(t)=0$

$$EZ(t)Z(s)=(f(\lambda)+g(\mu))^{-1}\left[\lambda(1-\lambda)(G(t, s)-2t)+2t(1-s)+\right.$$
$$\left.+(1-\lambda)^2t(1-s)+\mu(1-\mu)(H(t, s)-2t)+(1-\mu)^2t(1-s)\right],\quad t\leqq s.$$

Proof. Because the limit processes of $(n/g(\mu))^{1/2}(t-T_n(t;\mu))$ and $(n/f(\lambda))^{1/2}(t-K_n(t;\lambda))$ are independent, the proof follows from Theorem 3.1. The same way as in the preceding section we have

Corollary 5.2. *The process* $(n/(f(\lambda)+g(\mu)))^{1/2}(T_n(t;\mu)-K_n(t;\lambda))$ *converges weakly to the Brownian bridge for every* λ *and for every* μ $0\leqq\lambda, \mu\leqq 1$ *if and only if*

(i) $G(t, s)=k_1t(1-s)+2t$

$H(t, s)=k_2t(1-s)+2t,\quad t\leqq s$

where k_1, k_2 *are fixed real numbers with* $-2\leqq k_1, k_2\leqq 0$,

(ii) $f(\lambda)=k_1\lambda(1-\lambda)+1+(1-\lambda)^2$

$g(\mu)=k_2\mu(1-\mu)+1+(1-\mu)^2.$

Under the conditions of Corollary 5.2 the asymptotic solution of the reliability equation (5.1) is

$$M_{\lambda,\mu} \approx K\left(\left(k_1\lambda(1-\lambda)+k_2\mu(1-\mu)+(1-\lambda)^2+(1-\mu)^2+2\right)\log\frac{1}{\varepsilon}\Big/2n\right)^{1/2}.$$

The proportion of the minimal and maximal initial stock is again $(2/3)^{1/2}$.

6. Strong approximation of the process $X_n^{f(\lambda)}$. When talking about approximation of the empirical and quantile process by appropriate Gaussian processes, we think of constructing the latter on the probability space of the former so that they should be near to each other with probability one. This can be done if this probability space is rich enough in the sense that an infinite independent sequence of Wiener processes can be defined on it, which is also independent of the originally given i.i.d. sequence (cf. M. CSÖRGŐ—P. RÉVÉSZ [3] and KOMLÓS—MAJOR—TUSNÁDY [6]). It will be assumed that the underlying probability space is rich enough in this sense.

Theorem 6.1. *If X, Y are independent or $X=Y$ then one can define, for each n, a Brownian bridge $\{B_n(t), 0\leqq t\leqq 1\}$ such that we have*

$$P\left(\sup_{0\leqq t\leqq 1}|X_n^{f(\lambda)}(t)-B_n(t)| > K(\log n)^{3/4}n^{-1/4}\right) < Ln^{-2}$$

where $f(\lambda)=\lambda^2+2(1-\lambda)^2$ (X, Y are independent) or $f(\lambda)=1+(1-\lambda)^2$ ($X=Y$), and K, L are appropriate positive absolute constants.

Proof. Using the celebrated approximation result of KOMLÓS—MAJOR—TUSNÁDY [6] for the empirical process and that for the uniform quantile process of M. CSÖRGŐ and P. RÉVÉSZ [3], there exist Brownian bridges $B_n^{(1)}(t), B_n^{(2)}(t), B_n^{(3)}(t)$, which are independent (if X, Y are independent) or $B_n^{(1)}(t)=B_n^{(2)}(t)$ ($X=Y$) and they are near to $\beta_n(t; X)$, $\beta_n(t; Y)$, $q_n(t; Z^{-1})$. The representation (2.3) and the precise form of the Komlós—Major—Tusnády approximation easily gives

$$P\left(\sup_{0\leqq t\leqq 1}|X_n^{f(\lambda)}(t)-B_n(t)| > K(\log n)^{3/4}n^{-1/4}\right) \leqq$$

$$\leqq L_1 n^{-2}+P\left(\sup_{0\leqq t\leqq 1}\left|q_n(F_n(t; X); Z^{-1})-B_n^{(3)}(t)\right| > \frac{K}{3}(\log n)^{3/4}n^{-1/4}\right)$$

$$\leqq L_1 n^{-2}+P\left(\sup_{0\leqq t\leqq 1}\sup_{|s|\leqq(\log n/n)^{1/2}}|q_n(t+s; Z^{-1})-B_n^{(3)}(t)| > \frac{K}{3}(\log n)^{3/4}n^{-1/4}\right)+$$
$$+P\left(\sup_{0\leqq t\leqq 1}|F_n(t; X)-t| > (\log n/n)^{1/2}\right) \leqq$$

$$\leqq L_1 n^{-2}+P\left(\sup_{0\leqq t\leqq 1}|q_n(t; Z^{-1})-B_n^{(3)}(t)| > \frac{K}{6}(\log n)^{3/4}n^{-1/4}\right)+$$
$$+2P\left(\sup_{0\leqq t\leqq 1}\sup_{0\leqq s\leqq(\log n/n)^{1/2}}|B(t+s)-B(t)| > \frac{K}{6}(\log n)^{3/4}n^{-1/4}\right)+$$
$$+P\left(\sup_{0\leqq t\leqq 1}|F_n(t; X)-t| > (\log n/n)^{1/2}\right)$$

for any K, and a suitable $L_1>0$. Let now $K=\max(6A, 6\sqrt{30})$ where A is an appropriate constant to make the first probability here less then L_2n^{-2}, using the quantile process approximation with a suitable $L_2>0$. Using the lemma of DVORETZKY—KIEFER—WOLFOWITZ [2], the third probability is again smaller than L_3n^{-2}, where $L_3>0$ is some constant. Therefore the only problem now is to show that the second probability behaves the same way. Since $B(t)=W(t)-tW(1)$ with a standard Wiener process, this probability is not greater than

$$P\left(\sup_{0\le t\le 1}\sup_{0\le s\le(\log n/n)^{1/2}}|W(t+s)-W(t)| > \frac{1}{2}\sqrt{30}(\log n)^{3/4}n^{-1/4}\right)+$$

$$+P\left((\log n/n)^{1/2}|W(1)| > \frac{1}{2}\sqrt{30}(\log n)^{3/4}n^{-1/4}\right)\le$$

$$\le 40(n/\log n)^{1/2}n^{-5/2}+2(15\pi)^{-1}(n\log n)^{1/4}\exp\left(-15(n\log n)^{1/2}/2\right)\le L_4n^{-2}$$

where we have used the routine tail estimation of a normal variable, and, for the first term, Lemma 1 of M. CSÖRGŐ and P. RÉVÉSZ [4].

It follows (among others) under the conditions of Theorem 6.1 that

$$\sup_{-\infty<x<\infty}\left|P\left(\sup_{0\le t\le 1}X_n^{f(\lambda)}(t)<x\right)-P\left(\sup_{0\le t\le 1}B(t)<x\right)\right| = O\left((\log n)^{3/4}n^{-1/4}\right).$$

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Best approximation in Banach spaces with unconditional Schauder decompositions

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1. Introduction. The problem of the best approximation in Banach spaces with bases was initiated by NIKOLSKII [4]. SINGER [7, 8] carried out analogous study for the spaces with unconditional bases which has been further continued by RETHERFORD [5] and RETHERFORD and JAMES [6]. It has been pointed out that the results in these two settings are oftenly different. Motivated by this work and keeping in mind that a Banach space does not necessarily possess a basis as encountered by ENFLO [1], we consider Banach spaces with unconditional Schauder decomposition. In section 2, we give the notations and terminology. In section 3, the notions of NT-, NK- and NTK-norms have been defined in terms of the best approximation and a characterisation of each of these norms has been obtained. Also, it has been shown that every NT-norm is an NK-norm whereas the converse is not true is ascertained by giving a counterexample. Finally, it has been shown in section 4, that it is always possible to introduce an equivalent NTK-norm on a Banach space having an unconditional Schauder decomposition.

2. Notations and terminology. Let E be a Banach space, Z a linear subspace of E and x an element of E. An element $z_0 \in Z$ is a best approximation of x from Z provided

$$\|x-z_0\| = \inf\{\|x-z\|:\ z\in Z\}.$$

Thus, to every linear subspace Z of E and an element $x\in E$, there corresponds a bounded, closed and convex (possibly empty) set $B_Z(x)=\{z_0\in Z: z_0$ is a best approximation of $x\}$. We denote by π_Z the mapping of E into Z given by $\pi_Z(x)=z_0$ provided $B_Z(x)=\{z_0\}$.

A sequence (M_i) of nontrivial subspaces of E, is called a decomposition of E provided for each $x\in E$ there exists a unique sequence (x_i) such that $x_i\in M_i$

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and $x=\sum_{i=1}^{\infty} x_i$, the convergence being in the norm topology of E. It is possible to define for each i a projection $P_i\colon E\to M_i$ as $P_i(x)=x_i$. If each P_i is continuous, then (M_i) is called a Schauder decomposition, and we write (M_i, P_i). A decomposition (M_i, P_i) is said to be unconditional Schauder if it is Schauder with the property that $x=\sum_{i=1}^{\infty} x_{p(i)}$, for each permutation p of ω (the positive integers).

Let Σ denote the family of all finite subsets of ω. For $\sigma\in\Sigma$, let

$$L_\sigma = [\bigcup_{i\in\sigma} M_i] \quad\text{and}\quad L^\sigma = [\bigcup_{i\in\omega\setminus\sigma} M_i],$$

where the bracketed expressions denote the closed linear spans of the indicated sets. Also, we put

$$S_\sigma(x) = \sum_{i\in\sigma} P_i(x) \quad\text{and}\quad S^\sigma(x) = x - S_\sigma(x).$$

3. NT- and NK-norms. *Definition.* Let (M_i, P_i) be an unconditional Schauder decomposition of E. Then the norm $\|\ \|$ on E is called an *NT-norm* with respect to (M_i, P_i) if for every $x\in E$ and $\sigma\in\Sigma$, there exists a unique $y_0=\pi_{L_\sigma}(x)\in L_\sigma$, best approximation of x from L_σ, such that $\pi_{L_\sigma}(x)=S_\sigma(x)$; *NK-norm* with respect to (M_i, P_i) if for every $x\in E$ and $\sigma\in\Sigma$, there exists a unique $y_0=\pi_{L^\sigma}(x)\in L^\sigma$, best approximation of x from L^σ, such that $\pi_{L^\sigma}(x)=S^\sigma(x)$; and *NTK-norm* with respect to (M_i, P_i) if it is simultaneously an NT-norm and NK-norm with respect to this decomposition.

Now we characterise these norms as follows:

Theorem 1. *Let E be a Banach space with an unconditional Schauder decomposition (M_i, P_i). Then the norm on E is an*

(a) *NT-norm if and only if*

$$\|\sum_{i\in\omega\setminus\beta} x_i\| < \|\sum_{i\in\omega\setminus\alpha} x_i\|, \tag{3.1}$$

for every pair $\alpha, \beta\in\Sigma$ with $\alpha\subset\beta$ and every sequence $(x_i)_{i\in\omega\setminus\alpha}$ with $x_i\in M_i$ and $\sum_{i\in\beta\setminus\alpha} x_i\neq 0$, for which the series in (3.1) *are convergent*;

(b) *NK-norm if and only if*

$$\|\sum_{i\in\alpha} x_i\| < \|\sum_{i\in\beta} x_i\|, \tag{3.2}$$

for every pair $\alpha, \beta\in\Sigma$ with $\alpha\subset\beta$ and every finite sequence $(x_i)_{i\in\beta}$ with $x_i\in M$ and $\sum_{i\in\beta\setminus\alpha} x_i\neq 0$.

Proof. (a) Assume that the norm on E is an NT-norm. Let $\alpha, \beta\in\Sigma$ with $\alpha\subset\beta$ be arbitrary such that $\sum_{i\in\omega\setminus\alpha} x_i$ is convergent. Then

$$\pi_{L_\beta}(\sum_{i\in\omega\setminus\alpha} x_i) = S_\beta(\sum_{i\in\omega\setminus\alpha} x_i) = \sum_{i\in\beta\setminus\alpha} x_i,$$

and so

$$\|\sum_{i\in\omega\setminus\beta} x_i\| = \|\sum_{i\in\omega\setminus\alpha} x_i - \pi_{L_\beta}(\sum_{i\in\omega\setminus\alpha} x_i)\| < \|\sum_{i\in\omega\setminus\alpha} x_i\|,$$

which verifies the necessary part.

Conversely, for every $x=\sum_{i\in\omega} x_i\in E, \sigma\in\Sigma$ and $y=\sum_{i\in\sigma} y_i\in L_\sigma$ with $y\neq S_\sigma(x)$, we have (by (3.1) with $\beta=\sigma, \alpha=\emptyset$)

$$\|x-S_\sigma(x)\| = \|\sum_{i\in\omega\setminus\sigma} x_i\| < \|\sum_{i\in\omega\setminus\sigma} x_i - \sum_{i\in\sigma}(y_i-x_i)\| = \|x-y\|,$$

and thus the norm on E is an NT-norm.

(b) Assume that the norm on E is an NK-norm. Let $(x_i)_{i\in\beta}$ be a finite sequence and let $\alpha\subset\beta$ be such that $\sum_{i\in\beta\setminus\alpha} x_i\neq 0$. Then

$$\pi_{L^\alpha}(\sum_{i\in\beta} x_i) = \sum_{i\in\beta} x_i - S_\alpha(\sum_{i\in\beta} x_i) = \sum_{i\in\beta\setminus\alpha} x_i.$$

Hence

$$\|\sum_{i\in\alpha} x_i\| = \|\sum_{i\in\beta} x_i - \pi_{L^\alpha}(\sum_{i\in\beta} x_i)\| < \|\sum_{i\in\beta} x_i\|.$$

In order to establish the converse part, let $x=\sum_{i\in\omega} x_i$, $\sigma\in\Sigma$ and $y=\sum_{i\in\omega\setminus\sigma} y_i\in L^\sigma$ with $y\neq S^\sigma(x)$ be arbitrary. Then there exists in $\omega\setminus\sigma$ a smallest index i_0, such that $y_{i_0}\neq x_{i_0}$. Hence, applying (3.2) successively, we obtain

$$\|x-S^\sigma(x)\| = \|\sum_{i\in\sigma} x_i\| = \|\sum_{i\in\sigma} x_i - \sum_{\substack{i\in\omega\setminus\sigma\\ i<i_0}}(y_i-x_i)\| < \|\sum_{i\in\sigma} x_i - \sum_{\substack{i\in\omega\setminus\sigma\\ i\leqq i_0}}(y_i-x_i)\| \leqq \dots$$

$$\dots \leqq \|\sum_{i\in\sigma} x_i - \sum_{i\in\omega\setminus\sigma}(y_i-x_i)\| = \|x-y\|,$$

and thus the norm on E is an NK-norm. This completes the proof of the theorem.

Further, we give the relation between NT- and NK-norms.

Theorem 2. *Let E be a Banach space with an unconditional Schauder decomposition (M_i, P_i). Then every NT-norm with respect to (M_i) is an NK-norm (whence also an NTK-norm) with respect to (M_i).*

Proof. It follows by using (3.1) and (3.2).

The converse of Theorem 2 is not necessarily true. Consider for instance, the Banach space

$$c_0(\chi) = \{\bar{x} = (x_i)\colon x_i\in\chi,\ \operatorname*{Lim}_{i\to\infty} x_i = 0 \text{ in the norm of } \chi\},$$

the norm on $c_0(\chi)$ being given by $|(x_i)| = \sup_i \|x_i\|$, where $(\chi, \|\ \|)$ is any Banach space. On $c_0(\chi)$, define another norm $\|\ \|^*$ as:

$$\|(x_i)\|^* = \sup_{2\leqq n<\infty}\ \sup_{p\in\pi_{1,n}} \left(2^{-n}\|x_1\|/n + \sum_{i=2}^{\infty} 2^{-i}\|x_{p(i)}\|\right),$$

where $\pi_{1,n}$ denote the collection of all permutations of the set $\{2, 3, \ldots, n-1, n+1, n+2, \ldots\}$ and $p(n)=n$ for every $p\in\pi_{1,n}$. The norms $\| \ \|$ and $\| \ \|^*$ are equivalent since $\frac{1}{8}\|x\|\leqq\|x\|^*\leqq\frac{5}{8}\|x\|$. We observe that the sequence (N_i) with $N_i= =\{\delta_i^{x_i}: x_i\in\chi\}$, where $\delta_i^{x_i}$ we mean the sequence $(0, 0, \ldots, x_i, 0, \ldots)$, i.e. the ith entry in $\delta_i^{x_i}$ is x_i and all others are zero, forms an unconditional Schauder decomposition of $c_0(\chi)$ (see [2], p. 291 and [3], p. 95). Let $\alpha, \beta\in\Sigma, \alpha\subset\beta$ and $(\delta_i^{x_i})_{i\in\beta}$ with $\sum_{i\in\beta\setminus\alpha} \delta_i^{x_i}\neq 0$ be a finite sequence. Then

$$\|\sum_{i\in\alpha} \delta_i^{x_i}\|^* < \|\sum_{i\in\beta} \delta_i^{x_i}\|^*,$$

hence the norm $\| \ \|^*$ on $c_0(\chi)$ is an NK-norm. To show that $\| \ \|^*$ is not an NT-norm, it is enough to establish that

$$\left\|\sum_{m=1}^{\infty} \delta_m^{x_m}\right\|^* = \left\|\sum_{m=2}^{\infty} \delta_m^{x_m}\right\|^*, \tag{3.3}$$

with $x_m=\frac{1}{m}x$ for any $0\neq x\in\chi$.

Obviously

$$\left\|\sum_{m=1}^{\infty} \frac{1}{m}\delta_m^{x}\right\|^* \geqq \left\|\sum_{m=2}^{\infty} \frac{1}{m}\delta_m^{x}\right\|^*. \tag{3.4}$$

Furthermore, let $n\geqq 2$ be fixed. If, for a pair $i, i+j\in\{2, 3, \ldots, n-1, n+1, \ldots\}$ and a $p\in\pi_{1,n}$, we have $\frac{1}{p(i)}<\frac{1}{p(i+j)}$, then for the permutation $p'\in\pi_{1,n}$ defined by

$$p'(i) = p(i+j), \quad p'(i+j) = p(i), \quad p'(k) = p(k) \quad (k \neq i, \ i+j)$$

we have

$$\sum_{\substack{m=2\\ m\neq n}}^{\infty} \frac{\|x\|}{p'(m)2^m} > \sum_{\substack{m=2\\ m\neq n}}^{\infty} \frac{\|x\|}{p(m)2^m},$$

since

$$\frac{a}{2^i}+\frac{b}{2^{i+j}} > \frac{b}{2^i}+\frac{a}{2^{i+j}}, \quad \text{for} \ \ a > b \geqq 0.$$

Consequently, for every $n\geqq 2$ and $p\in\pi_{1,n}$, we have

$$\frac{\|x\|}{n2^n}+\sum_{\substack{m=2\\ m\neq n}}^{\infty} \frac{\|x\|}{p(m)2^m} \leqq \frac{\|x\|}{n2^n}+\sum_{\substack{m=2\\ m\neq n}}^{\infty} \frac{\|x\|}{m2^m} = \sum_{m=2}^{\infty} \frac{\|x\|}{m2^m} = \sup_{2\leqq n<\infty} \sum_{\substack{m=2\\ m\neq n}}^{\infty} \frac{\|x\|}{m2^m} \leqq$$

$$\leqq \sup_{2\leqq n<\infty} \sup_{\tau\in\pi_{1,n}} \sum_{\substack{m=2\\ m\neq n}}^{\infty} \frac{\|x\|}{\tau(m)2^m} = \left\|\sum_{m=2}^{\infty} \frac{1}{m}\delta_m^{x}\right\|^*,$$

which together with (3.4) implies (3.3).

4. An NTK-norm. If E is a Banach space with an unconditional Schauder decomposition (M_i, P_i) then it is always possible to introduce on E an NTK-norm equivalent to the original norm on E. Consider for instance

$$\|x\|_{\mathrm{NTK}} = \sum_{i\in\omega} \|P_i(x)\| \, 2^{-i} + \sup_{\sigma\in\omega} \Big\|\sum_{i\in\sigma} P_i(x)\Big\|.$$

This clearly defines a norm on E, and is equivalent to the original norm on E which follows from

$$\|x\| \leqq \|x\|_{\mathrm{NTK}} \leqq \max_{1\leqq i<\infty} \|P_i(x)\| + \sup_{\sigma\in\omega} \Big\|\sum_{i\in\sigma} P_i(x)\Big\| \leqq 3K\|x\|.$$

Finally, let $\alpha, \beta\in\Sigma$ with $\alpha\subset\beta$ and $(x_i)_{i\in\omega\setminus\beta}$ with $\sum_{i\in\beta\setminus\alpha} x_i \neq 0$, $x_i\in M_i$, $i\in\omega$, be such that $\sum_{i\in\omega\setminus\beta} x_i$ converges. Then, we have $\omega\setminus\alpha=(\omega\setminus\beta)\cup(\beta\setminus\alpha)$, hence

$$\Big\|\sum_{i\in\omega\setminus\beta} x_i\Big\|_{\mathrm{NTK}} = \sum_{i\in\omega\setminus\beta} \|x_i\| \, 2^{-i} + \sup_{\sigma\in\Sigma} \Big\|\sum_{i\in\sigma\cap(\omega\setminus\beta)} x_i\Big\| <$$
$$< \sum_{i\in\omega\setminus\alpha} \|x_i\| \, 2^{-i} + \sup_{\sigma\in\Sigma} \Big\|\sum_{i\in\sigma\cap(\omega\setminus\alpha)} x_i\Big\| = \Big\|\sum_{i\in\omega\setminus\alpha} x_i\Big\|_{\mathrm{NTK}}.$$

Thus by Theorem 1, $\| \; \|_{\mathrm{NTK}}$ is an NT-norm and hence an NTK-norm by Theorem 2.

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On p-weak contractions

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H. BERCOVICI and D. VOICULESCU [1] defined the algebraic adjoint of operators belonging to the class $I+\mathscr{C}_1$. In Sec. 2 we extend this notion to the operators of class $I+\mathscr{C}_p$, where $p\geqq 1$ is an arbitrary integer. In Sec. 3 we study contractions T such that $\sigma(T)\neq D^-$ and $I-T^*T\in\mathscr{C}_p$, where $p\geqq 1$ is an arbitrary real number. These contractions will be called *p-weak*. We show that their characteristic functions have (generally unbounded) scalar multiples. With the aid of this we characterize in Sec. 4 and 5 the spectra of p-weak contractions and some $C_{\cdot 1}$ contractions.

In [1] it was proved that a C_0 contraction is a weak if and only if its Jordan-model is a weak contraction. In Sec. 6 we study the validity of this statement for p-weak contractions.

1. Preliminaries

We shall consider separable Hilbert spaces over the complex field $\mathbf{C}$.

If A is a compact operator on the Hilbert space $\mathfrak{K}$, then $|A|=(A^*A)^{\frac{1}{2}}$ is a compact selfadjoint operator. So there exist a decreasing sequence of positive numbers $\{s_i\}$ (the *s-numbers of* A) and an orthonormal system $\{\varphi_i\}$ such that $\lim s_i=0$ and $|A|=\sum_i s_i\langle\cdot,\varphi_i\rangle\varphi_i$. If $p\geqq 1$ is an arbitrary real number, then the *Schatten class* $\mathscr{C}_p(\mathfrak{K})$ is the set of compact operators A such that $\sum_i (s_i(A))^p<\infty$. It can be shown that $\mathscr{C}_p(\mathfrak{K})$ is a two-sided self-adjoint ideal in $\mathscr{L}(\mathfrak{K})$ and the function $\|A\|_p:=[\sum_i s_i^p]^{\frac{1}{p}}$ is a Banach-norm in $\mathscr{C}_p$.

For arbitrary operators $A, C\in\mathscr{L}(\mathfrak{K})$ and $B\in\mathscr{C}_p(\mathfrak{K})$ we have $\|ABC\|_p\leqq\leqq\|A\|\,\|B\|_p\|C\|$. If $A_j\in\mathscr{C}_{p_j}$ $(j=1, \dots, n)$ and $\sum_{j=1}^n p_j^{-1}\leqq 1$ then $A=A_1\dots A_n\in\mathscr{C}_p$,

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where $p^{-1}=\sum_{j=1}^{n} p_j^{-1}$; moreover, $\|A\|_p \leqq \|A_1\|_{p_1} \dots \|A_n\|_{p_n}$. Let $\{B_n\}$ be a sequence of operators tending to the operator B in the strong sense and let A be an operator from $\mathscr{C}_p(\mathfrak{K})$. Then $\lim_{n\to\infty} \|B_nA-BA\|_p=\lim_{n\to\infty} \|AB_n-AB\|_p=\lim_{n\to\infty} \|B_nAB_n-BAB\|_p=0$.

The operators $A\in\mathscr{C}_1(\mathfrak{K})$ are those with finite trace, i.e. for which $\sum_{i=1}^{\infty} \langle A\varphi_i, \varphi_i\rangle$ is convergent for every orthonormal basis $\{\varphi_i\}$. This sum is independent from the choice of the basis and is called the *trace* of A, and is denoted by $\operatorname{tr} A$ or $\operatorname{sp} A$. The following properties hold. For every $A_1, A_2\in\mathscr{C}_1$ and $c_1, c_2\in\mathbf{C}$ we have $\operatorname{sp}(c_1A_1+c_2A_2)=c_1 \operatorname{sp} A_1+c_2 \operatorname{sp} A_2$. If $AB, BA\in\mathscr{C}_1$ and A or B is compact, then $\operatorname{sp}(AB)=\operatorname{sp}(BA)$. If $A\in\mathscr{C}_1$, then $\operatorname{sp} A^*=\overline{\operatorname{sp} A}$ and $|\operatorname{sp} A|\leqq\|A\|_1$.

Let us assume that the operator A has the form $A=I-X$ where $X\in\mathscr{C}_p$ ($p\geqq 1$ integer). Let $\{\lambda_j\}$ be the sequence of the characteristic values of X taking them according to their algebraic multiplicities and let $\{s_j\}$ be the sequence of the s-numbers of X. It can be proved that $\sum_j |\lambda_j|^p \leqq \sum_j s_j^p$, so $\sum_j |\lambda_j|^p<\infty$. Therefore we can define the *p-regulated determinant of* A by

$$\overset{(p)}{\det} A := \prod_j \left[(1-\lambda_j)\exp\left(\sum_{k=1}^{p-1} \frac{1}{k}\lambda_j^k\right)\right].$$

If $A\in I+\mathscr{C}_1$, then $\overset{(p)}{\det} A=(\det A)\exp\left[\sum_{k=1}^{p-1} \operatorname{sp}(I-A)^k\right]$, where $\det A=\overset{(1)}{\det} A= =\prod_j (1-\lambda_j)$. The function $\overset{(p)}{\det}(\cdot)$ is continuous in the following sense. If A, A_n are operators from $I+\mathscr{C}_p$ $(n=1, 2, \dots)$ and $\lim_{n\to\infty}\|A_n-A\|_p=0$, then $\lim_{n\to\infty} \overset{(p)}{\det} A_n= =\overset{(p)}{\det} A$.

For a detailed discussion of these facts see [2], ch. II, III, IV.

Let A be an arbitrary operator on the finite dimensional Hilbert space $\mathfrak{K}$, having the matrix $[a_{i,j}]_{i,j=1}^n$ in the orthonormal basis $\{e_i\}_{i=1}^n$. Let us denote by $b_{i,j}$ the determinant, multiplied by $(-1)^{i+j}$, of the matrix obtained from the matrix $[a_{i,j}]_{i,j=1}^n$ by deleting its ith column and jth line $(1\leqq i, j\leqq n)$. The *algebraic adjoint* A^{Ad} *of* A is defined as the operator having the matrix $[b_{i,j}]_{i,j=1}^n$ in the basis $\{e_i\}_{i=1}^n$. It can be shown that this definition does not depend on the choice of the basis $\{e_i\}_{i=1}^n$. For details we refer to [1], § 5.

For any two (separable) Hilbert spaces $\mathfrak{E}, \mathfrak{E}_*$ the operator-valued Hardy space $H^\infty(\mathscr{L}(\mathfrak{E}, \mathfrak{E}_*))$ is the set of all bounded, $\mathscr{L}(\mathfrak{E}, \mathfrak{E}_*)$-valued analytic functions in the unit disc $D=\{z\in\mathbf{C} \,|\, |z|<1\}$. A function $\Theta\in H^\infty(\mathscr{L}(\mathfrak{E}, \mathfrak{E}_*))$ is contractive if $\|\Theta(z)\|\leqq 1$, $z\in D$. It is purely contractive if moreover $\|\Theta(0)f\|<\|f\|$ for every $f\in\mathfrak{E}, f\neq 0$. We say that two functions $\Theta_i\in H^\infty(\mathscr{L}(\mathfrak{E}_i, \mathfrak{E}_{*i}))$ $(i=1, 2)$ *coincide* if there are unitary operators $U\colon \mathfrak{E}_1\to\mathfrak{E}_2$, $V\colon \mathfrak{E}_{*1}\to\mathfrak{E}_{*2}$ such that $\Theta_2(\lambda)U=V\Theta_1(\lambda)$ for all $\lambda\in D$. A function $\Theta\in H^\infty(\mathscr{L}(\mathfrak{E}, \mathfrak{E}_*))$ is *outer* if it has dense range as an element of $\mathscr{L}(H^2(\mathfrak{E}), H^2(\mathfrak{E}_*))$. The function $\Theta^\sim$ is defined by $\Theta^\sim(z)=(\Theta(\bar z))^*$, $z\in D$.

If $\Theta\in H^\infty(\mathscr{L}(\mathfrak{E},\mathfrak{E}_*))$ is purely contractive, then $S(\Theta)$ is the operator acting on the Hilbert space

$$\mathbf{H} = [H^2(\mathfrak{E}_*)\oplus\overline{\Delta L^2(\mathfrak{E})}]\ominus\{\Theta w\oplus\Delta w | w\in H^2(\mathfrak{E})\},$$

where $\Delta(t)=(I-\Theta(e^{it})^*\Theta(e^{it}))^{\frac{1}{2}}$, and defined by

$$S(\Theta)^*(u_*\oplus v) = e^{-it}(u_*(e^{it})-u_*(0))\oplus e^{-it}v(t),$$

$u_*\oplus v\in\mathbf{H}$.

If T is a contraction on the Hilbert space $\mathfrak{H}$, then $D_T=(I-T^*T)^{\frac{1}{2}}$, $D_{T^*}=(I-TT^*)^{\frac{1}{2}}$ will be called the *defect operators,* and $\mathfrak{D}_T=(D_T\mathfrak{H})^-$, $\mathfrak{D}_{T^*}=(D_{T^*}\mathfrak{H})^-$ the *defect spaces of* T. The *characteristic function* of the contraction T is the puraly contractive function $\Theta_T\in H^\infty(\mathscr{L}(\mathfrak{D}_T,\mathfrak{D}_{T^*}))$ defined by

$$\Theta_T(\lambda) = [-T+\lambda D_{T^*}(I-\lambda T^*)^{-1}D_T]|\mathfrak{D}_T,\quad \lambda\in D.$$

A contraction T on $\mathfrak{H}$ is *completely non-unitary* (c.n.u.) if for no non-zero reducing subspace $\mathfrak{L}$ for T is $T|\mathfrak{L}$ a unitary operator. For these facts see [3], ch. V, VI.

If $\{m_n\}_n$ is a sequence of inner functions such that m_{n+1} divides m_n for all n, then we call the operator $\bigoplus\limits_n S(m_n)$ a *Jordan-operator.* It was proved in [4] that for every C_0 operator T (cf. [3], ch. III) there exists a unique Jordan-operator S, the *Jordan-model of* T, such that T and S are quasi-similar.

2. Algebraic adjoints of operators of class $I+\mathscr{C}_p$

In this section we extend the notion of algebraic adjoint defined by H. Bercovici and D. Voiculescu [1] in the case $p=1$ to the operators of class $I+\mathscr{C}_p$, where $p\geqq 1$ is an arbitrary integer. The definition will be introduced, as in [1], in three steps. Firstly we treat the finite dimensional case, after that the case when the operator belongs to the class $I+\mathscr{F}$ ($\mathscr{F}$ denotes the class of operators having finite rank) and at last the case when the operator is taken from the class $I+\mathscr{C}_p$.

2.1. The finite dimensional case. In this section let $\mathfrak{K}$ be a finite dimensional Hilbert space, $\dim\mathfrak{K}=n$.

Definition 2.1. If $A\in\mathscr{L}(\mathfrak{K})$ and $p\geqq 2$ is an integer, then the *p-regulated algebraic adjoint of* A is the operator defined by

$$\overset{(p)}{A^{\mathrm{Ad}}}:= A^{\mathrm{Ad}}\exp\left[\sum_{k=1}^{p-1}\frac{1}{k}\operatorname{sp}(I-A)^k\right].$$

Proposition 2.2.

(i) $A\overset{(p)}{A^{\mathrm{Ad}}}=\overset{(p)}{A^{\mathrm{Ad}}}A=(\overset{(p)}{\det}A)I$;

(ii) *If* $\{e_i\}_{i=1}^n$ *is an orthonormal basis in* $\mathfrak{K}$, *and* $1\leqq i,j\leqq n$, *then*

$$\langle \overset{(p)}{A^{\mathrm{Ad}}}e_i, e_j\rangle=(\overset{(p)}{\det}A_{ij})\exp\left[\mathrm{sp}\left(q_p(A, U_{i,j}, P_j)P_j\right)\right],$$

where $P_j=\langle\cdot, e_j\rangle e_j$, $U_{i,j}=\langle\cdot, e_j\rangle e_i$, $A_{i,j}=U_{i,j}+A(I-P_j)$ *and* $q_p(x_1, x_2, x_3)$ *is a polynomial in its non-commuting variables (depending only on* p*).*

(iii) *There exist constants* $D_p>0$, $C_p>0$ *and* $\gamma_p>0$ *(*γ_p *being an integer) depending only on* p *such that*

$$\|\overset{(p)}{A^{\mathrm{Ad}}}\|\leqq D_p\exp\left[C_p\|I-A\|_p^{\gamma_p}\right].$$

Property (i) immediately follows from the definition. As for property (ii), we have

$$\langle \overset{(p)}{A^{\mathrm{Ad}}}e_i, e_j\rangle=\langle A^{\mathrm{Ad}}e_i, e_j\rangle\exp\left[\sum_{k=1}^{p-1}\frac{1}{k}\,\mathrm{sp}\,(I-A)^k\right]=$$

$$=(\overset{(p)}{\det}A_{i,j})\exp\left[\mathrm{sp}\left(\sum_{k=1}^{p-1}\frac{1}{k}\left((I-A)^k-(I-A_{i,j})^k\right)\right)\right]=$$

$$=(\overset{(p)}{\det}A_{i,j})\exp\left[\mathrm{sp}\left(\sum_{k=1}^{p-1}\frac{1}{k}\left((I-A)^k-[(I-A)+(A-U_{i,j})P_j]^k\right)\right)\right]=$$

$$=(\overset{(p)}{\det}A_{i,j})\exp\left[\mathrm{sp}\left(q_p(A, U_{i,j}, P_j)P_j\right)\right].$$

For proving property (iii) we need the next lemmas.

Lemma 2.3. *For every integer* $p\geqq 2$ *there exists a constant* $C_p^*>0$ *such that for all* $\lambda\in\mathbf{C}$ *we have*

$$\left|f_p(\lambda)=(1-\lambda)\exp\left[\lambda+\frac{1}{2}\lambda^2+\ldots+\frac{1}{p-1}\lambda^{p-1}\right]\right|\leqq\exp\left[C_p^*|\lambda|^p\right].$$

Proof. In the case $|\lambda|\leqq\frac{1}{2}$ we have

$$|f_p(\lambda)|=\left|\exp\left[\log^*(1-\lambda)+\lambda+\frac{1}{2}\lambda^2+\ldots+\frac{1}{p-1}\lambda^{p-1}\right]\right|=\left|\exp\left[-\sum_{n=p}^{\infty}\frac{\lambda^n}{n}\right]\right|\leqq\exp\left[|\lambda|^p\right].$$

If $|\lambda|\geqq p$, then $|f_p(\lambda)|\leqq\exp\left[2|\lambda|+\frac{1}{2}|\lambda|^2+\ldots+\frac{1}{p-1}|\lambda|^{p-1}\right]\leqq\exp\left[p|\lambda|^{p-1}\right]\leqq$ $\leqq\exp\left[|\lambda|^p\right]$. Now there exists a constant M_p, such that $|f_p(\lambda)|\leqq M_p$ if $\frac{1}{2}\leqq|\lambda|\leqq p$.

Choosing $C_p' \geqq 2^p \ln M_p$ we have $|f_p(\lambda)| \leqq M_p \leqq \exp\left[C_p'\left(\frac{1}{2}\right)^p\right] \leqq \exp[C_p'|\lambda|^p]$ when $\frac{1}{2} \leqq |\lambda| \leqq p$. Therefore $C_p^* = \max\{1, C_p'\}$ will be suitable.

Lemma 2.4. *There exist constants $D_{p,N}$ and $C_{p,N}$ such that for every normal operator A we have*

$$\|\overset{(p)}{A^{\mathrm{Ad}}}\| \leqq D_{p,N} \exp[C_{p,N}\|I-A\|_p^p]. \tag{2.1}$$

Proof. There exist an orthonormal basis $\{e_i\}_{i=1}^n$ and complex numbers $\{\lambda_i\}_{i=1}^n$ such that $I-A=\sum_{i=1}^n \lambda_i\langle\cdot, e_i\rangle e_i$. Then denoting by $A^{\wedge(n-1)}$ the exterior product, taking A $(n-1)$ times, we have

$$\|\overset{(p)}{A^{\mathrm{Ad}}}\| = \|A^{\wedge(n-1)}\|\left|\exp\left[\sum_{k=1}^{p-1}\frac{1}{k}\operatorname{sp}(I-A)^k\right]\right| =$$

$$= \prod_{\substack{i=1\\ i\neq i_0}}^{n}\left|(1-\lambda_i)\exp\left[\sum_{k=1}^{p-1}\frac{1}{k}\lambda_i^k\right]\right|\exp\left[\operatorname{Re}\sum_{k=1}^{p-1}\frac{1}{k}\lambda_{i_0}^k\right]$$

for some index i_0. By virtue of Lemma 2.3 we have

$$\|\overset{(p)}{A^{\mathrm{Ad}}}\| \leqq \exp\left[C_p^*\sum_{i=1}^n|\lambda_i|^p\right]\cdot\exp\left[\operatorname{Re}\left(\sum_{k=1}^{p-1}\frac{1}{k}\lambda_{i_0}^k\right)-C_p^*|\lambda_{i_0}|^p\right] \leqq D_p^*\exp[C_p^*\|I-A\|_p^p],$$

where D_p^* is an upper bound of the second factor, when λ_{i_0} alters in $\mathbf{C}$. $D_{p,N}=D_p^*$ and $C_{p,N}=C_p^*$ will be suitable constants.

Lemma 2.5. *For arbitrary operators X, Y and integer $p\geqq 2$ $R_p(X, Y)$ denotes the operator given by*

$$R_p(X, Y) = \sum_{k=1}^{p-1}\frac{1}{k}[(X+Y-XY)^k - X^k - Y^k]. \tag{2.2}$$

Let $R_p'(X, Y)$ be the polynomial obtained from $R_p(X, Y)$ omitting the terms of degree less than p. Then

$$\operatorname{sp} R_p'(X, Y) = \operatorname{sp} R_p(X, Y). \tag{2.3}$$

Proof. We can assume that $p>2$. Obviously

$$R_p(X, Y) = R_p'(X, Y) + \sum_{q=2}^{p-1} K_q(X, Y), \tag{2.4}$$

where $K_q(X, Y)$ represents the homogeneous part of degree q of R_p. So it will be enough to prove that

$$\operatorname{sp} K_q(X, Y) = 0 \quad \text{for} \quad q = 2, \ldots, p-1. \tag{2.5}$$

Let q be an arbitrary integer such that $2 \leqq q \leqq p-1$. All terms in K_q contain both X and Y.

Let T be the mapping on the formal products containing q factors of operators from $\mathscr{L}(\mathfrak{K})$ such that

$$T(A_1 \dots A_q) := A_2 A_3 \dots A_q A_1.$$

If $i \geqq 1, j \geqq 1$ and $i+j=q$, then let us denote $\mathscr{L}_{i,j}$ the set of $S(X, Y)$'s, where $S(X, Y)$ is a product containing X i-times and Y j-times. We call $S(X, Y)$ *equivalent to* $S'(X, Y)$ if there exists an integer $r \geqq 0$ such that $T^r(S(X, Y)) = S'(X, Y)$. It is clear that this is an equivalence relation on $\mathscr{L}_{i,j}$. If $S(X, Y)$ is equivalent to $S'(X, Y)$, then $\operatorname{sp} S(X, Y) = \operatorname{sp} S'(X, Y)$. Therefore to verify (2.5) it is enough to prove that in any case $i \geqq 1, j \geqq 1, i+j=q$, taking an arbitrary equivalence class of $\mathscr{L}_{i,j}$ the sum of the coefficients in K_q of the products belonging to this equivalence class is 0.

Let S be an arbitrary element in $\mathscr{L}_{i,j}$. We compute the coefficient of S in K_q. The factor Y is called *essential* in S if an X factor precedes it. Let us suppose that the number of essential Y's in S is j_0 $(0 \leqq j_0 \leqq j)$. We can get $S(X, Y)$ in K_q such that (XY) occurs s-times $(0 \leqq s \leqq j)$ as a factor only from $\frac{1}{q-s} \cdot (X+Y-XY)^{q-s}$. Since the number of the factors (XY) is j_0, the coefficient derived so is $\binom{j_0}{s}(-1)^s \frac{1}{q-s}$. We get that the coefficient of $S(X, Y)$ in K_q is

$$\sum_{s=0}^{j_0} (-1)^s \binom{j_0}{s} \frac{1}{q-s}. \tag{2.6}$$

We denote by $\hat{S}$ the equivalence class of S in $\mathscr{L}_{i,j}$. There exists a least positive integer r such that $T^r S = S$. We infer that r devides q, and $\hat{S} = \{S, TS, \dots, T^{r-1}S\}$. We may assume that the first factor in S is X, so $j_0 \geqq 1$. The number of essential Y's in $T^l S$ is $(j_0 - 1)$ if the first factor of $T^l S$ is an operator Y which has occurred in S as an essential factor. Otherwise this number is j_0. Therefore there exist $\frac{r}{q} j_0$ elements in $\hat{S}$ such that they have (j_0-1) essential Y's, and there exist $\frac{r}{q}(q-j_0)$ elements in $\hat{S}$ such that they have j_0 essential Y's. By virtue of (2.6) the sum of the coefficients of the elements of $\hat{S}$ in K_q is

$$\frac{r}{q} j_0 \sum_{s=0}^{j_0-1} (-1)^s \binom{j_0-1}{s} \frac{1}{q-s} + \frac{r}{q}(q-j_0) \sum_{s=0}^{j_0} (-1)^s \binom{j_0}{s} \frac{1}{q-s}. \tag{2.7}$$

A short computation shows that this sum is 0.

The Lemma is proved.

Lemma 2.6. *Let A be an arbitrary operator and $A=U|A|$ its polar decomposition. Let us denote $X=I-U$, $Y=I-|A|$ and $Z=I-A$. Then*

$$\|X\|_p \leqq \|Z\|_p^2+3\|Z\|_p \quad \text{and} \quad \|Y\|_p \leqq \|Z\|_p^2+2\|Z\|_p. \tag{2.8}$$

Proof. Because $|1-\sqrt{\lambda}|\leqq|1-\lambda|$ $(\lambda\geqq 0)$ we have

$$\|Y\|_p = \|I-|A|\|_p \leqq \|I-A^*A\|_p. \tag{2.9}$$

On the other hand from the identity $I-A^*A = -(I-A^*)(I-A)+(I-A)+(I-A^*)$ we can derive

$$\|I-A^*A\|_p \leqq \|I-A\|_p^2+2\|I-A\|_p. \tag{2.10}$$

The second relation of (2.8) follows from (2.9) and (2.10). The first one can be got from this and the inequality $\|X\|_p\leqq\|Y\|_p+\|Z\|_p$ derived from the equation $X^*=Y-U^*Z$.

We can now prove the property (iii) in Proposition 2.2. Let A be an arbitrary operator and let X, Y, Z be defined as in the Lemma 2.6. A short computation yields

$$\overset{(p)}{A^{\mathrm{Ad}}} = \overset{(p)}{|A|^{\mathrm{Ad}}}\,\overset{(p)}{U^{\mathrm{Ad}}} \exp[\operatorname{sp} R_p(X, Y)], \tag{2.11}$$

where $R_p(X, Y)$ is given by (2.2). Applying Lemmas 2.4 and 2.5 we infer that there exists a polynomial r_p with positive coefficients such that

$$\|\overset{(p)}{A^{\mathrm{Ad}}}\| \leqq \exp[r_p(\|X\|_p, \|Y\|_p)]. \tag{2.12}$$

On account of Lemma 2.6 property (iii) follows.

2.2. Definition and properties in the case $A\in I+\mathscr{F}$. There exists a decomposition $\mathfrak{R}=\mathfrak{R}_1\oplus\mathfrak{R}_2$ reducing A such that $\dim\mathfrak{R}_1=n<\infty$ and A has the form $A=A_1\oplus I_2$.

Definition 2.7. $\overset{(p)}{A^{\mathrm{Ad}}}:=\overset{(p)}{A_1^{\mathrm{Ad}}}\oplus(\overset{(p)}{\det} A_1)I_2$.

Proposition 2.8. *The properties in Proposition* 2.2 *hold.*

Proof. (i) is evident. (iii) follows from the same property of Proposition 2.2 and the inequality $|\overset{(p)}{\det} A_1| = \left|\prod_{i=1}^{n}(1-\lambda_i)\exp\left[\lambda_i+\frac{1}{2}\lambda_i^2+\dots+\frac{1}{p-1}\lambda_i^{p-1}\right]\right| \leqq$ $\leqq \exp\left[C_p^*\sum_{i=1}^{n}|\lambda_i|^p\right]=\exp[C_p^*\|I_1-A_1\|_p^p]$, where λ_i's are the characteristic roots of A_1. (We have used Lemma 2.3.)

We prove (ii) firstly in the special case when the basis $\{e_i\}$ is such that $e_1\oplus\dots\oplus e_n=\mathfrak{R}_1$. It can be easily verified that for every $1\leqq i, j$

$$\langle \overset{(p)}{A^{\mathrm{Ad}}} e_i, e_j\rangle = (\overset{(p)}{\det} A_{i,j})\exp\left[\operatorname{sp}\left(\sum_{k=1}^{p-1}\frac{1}{k}((I-A)^k-(I-A_{i,j})^k)\right)\right].$$

Then property (ii) follows as in the proof of Proposition 2.2. Now we can easily see that the definition of $\overset{(p)}{A^{\mathrm{Ad}}}$ does not depend on the decomposition $\mathfrak{K}=\mathfrak{K}_1\oplus\mathfrak{K}_2$. Therefore property (ii) is fulfilled in general.

2.3. Definition and properties when $A\in I+\mathscr{C}_p$. There exists a sequence of operators $\{A_n\}_n$ such that $A_n\in I+\mathscr{F}$ for all n, and $\lim_n \|A_n-A\|_p=0$. Let $\{e_k\}$ be an orthonormal basis in $\mathfrak{K}$, and $1\leq i,j$. Since $\lim_n \|A_{i,j}^{(n)}-A_{i,j}\|_p=0$ also holds, we infer $\lim_n \overset{(p)}{\det} A_{i,j}^{(n)}=\overset{(p)}{\det} A_{i,j}$. On the other hand,

$$\lim_n \|q_p(A_n,\, U_{i,j},\, P_j)P_j-q_p(A,\, U_{i,j},\, P_j)P_j\|_1=0$$

for $\lim_n \|A_n-A\|=0$ and rank $P_j=1$. So we can write $\lim_n \operatorname{sp}\big(q_p(A_n,\, U_{i,j},\, P_j)P_j\big)= =\operatorname{sp}\big(q_p(A,\, U_{i,j},\, P_j)P_j\big)$. Therefore by virtue of property (ii) of Proposition 2.8 we get

$$\lim_n \langle \overset{(p)}{A_n^{\mathrm{Ad}}} e_i,\, e_j\rangle=\overset{(p)}{\det} A_{i,j}\exp\big[\operatorname{sp}\big(q_p(A,\, U_{i,j}, P_j)P_j\big)\big].$$

Regarding property (iii) of Proposition 2.8 we see that $\{\|\overset{(p)}{A_n^{\mathrm{Ad}}}\|\}_n$ is bounded. So the operator sequence $\{\overset{(p)}{A_n^{\mathrm{Ad}}}\}_n$ is weakly convergent.

Definition 2.9. $\overset{(p)}{A^{\mathrm{Ad}}}:=\lim_n \overset{(p)}{A_n^{\mathrm{Ad}}}$, where the limit exists in the sense of weak operator convergence. We call $\overset{(p)}{A^{\mathrm{Ad}}}$ the *p-regulated algebraic adjoint of* A.

Theorem 2.10. *If* $A\in I+\mathscr{C}_p$ *is an arbitrary operator, then the properties of Proposition* 2.2 *are satisfied.*

Proof. We can infer these properties from the definition and the corresponding properties of Proposition 2.8.

Remark 2.11. We can define similarly p-regulated algebraic adjoints of higher order of operators belonging to $I+\mathscr{C}_p$.

3. p-weak contractions

In this section $p\geq 1$ is an arbitrary real number.

Definition 3.1. A contraction T will be called a *p-weak contraction* if its spectrum $\sigma(T)$ does not fill the unit disc D and $I-T^*T$ belongs to the class $\mathscr{C}_p$.

Remark 3.2. We can easily see that if $I-T^*T$ is compact then there exist a maximal partial isometry U and a compact operator X such that $T=U+X$.

By reason of the properties of semi-Fredholm operators (cf. [5]) we infer that if $\sigma_p(T)\cap D\neq D$ and $\sigma_p(T^*)\cap D\neq D$ then $\sigma_p(T)\cap D=\overline{\sigma_p(T^*)}\cap D=\sigma(T)\cap D$ and $\sigma(T)\cap D$ consists of isolated points in D. ($\sigma_p(T)$ denotes the point-spectrum of T.) Therefore we can state that a contraction T is p-weak if and only if $\sigma_p(T)$ and $\sigma_p(T^*)$ does not fill the unit disc D and $I-T^*T\in\mathscr{C}_p$.

Definition 3.1 is a generalization of the concept of weak contractions (case $p=1$). Several properties carry over to this case also.

Theorem 3.3. *If T is a p-weak contraction, then so are*
(i) $T_a=(T-aI)(I-\bar{a}T)^{-1}$, *where* $a\in D$;
(ii) T^*;
(iii) $T|\mathfrak{L}$, *where $\mathfrak{L}$ is an invariant subspace for T and $\sigma(T|\mathfrak{L})\neq D^-$.*

Proof. $\mathscr{C}_p$ being a two-sided ideal we see that $I-T_a^*T_a=(1-|a|^2)(I-aT^*)^{-1}\cdot\cdot(I-T^*T)(I-\bar{a}T)^{-1}\in\mathscr{C}_p$. Properties (ii) and (iii) follow as in [3], ch. VIII.

Theorem 3.4. *If T is a p-weak contraction then there exists a contractive analytic function $\Theta_0\in H^\infty(\mathscr{L}(\mathfrak{E}))$ coinciding with the characteristic function of T such that $\Theta_0(\lambda)\in I+\mathscr{C}_p(\mathfrak{E})$ for all $\lambda\in D$. All such function is of the form $U\Theta_0$ (regarding unitary equivalence), where U is an arbitrary unitary operator belonging to the class $I+\mathscr{C}_p$. Moreover Θ_0 can be chosen such that for all unitary operator $U\in I+\mathscr{C}_p(\mathfrak{E})$ and $\lambda\in D$ we have*

$$\|I-\Theta(\lambda)\|_p\leqq\|D_T^2\|_p(1-|\lambda|)^{-1}+\|I-U\|_p,$$

where $\Theta=U\Theta_0$.

Proof. Let $a\in D\setminus\sigma(T)$ and $T_a=(T-aI)(I-\bar{a}T)^{-1}$. We have $D_{T_a}^2=\\=\sum_n\mu_n\langle\cdot,\varphi_n\rangle\varphi_n$, $D_{T_a^*}^2=\sum_n\mu_n\langle\cdot,\psi_n\rangle\psi_n$ where $\{\varphi_n\}$, $\{\psi_n\}$ are orthonormal systems (cf. [3], ch. VIII). The operator $U_a\in\mathscr{L}(\mathfrak{D}_{T_a},\mathfrak{D}_{T_a^*})$ defined by $U_a\varphi_n=-\psi_n$ will be unitary and $\Theta_a(\lambda)=U_a^*\Theta_{T_a}(\lambda)\in I+\mathscr{C}_p(\mathfrak{D}_{T_a})$ for all $\lambda\in D$. (Θ_T and Θ_{T_a} are the characteristic functions of T and T_a respectively.)

$$\|I-\Theta_a(\lambda)\|_p\leqq\|(I+U_a^*T_a)|\mathfrak{D}_{T_a}\|_p+\|\lambda U_a^*D_{T_a^*}(I-\lambda T_a^*)^{-1}D_{T_a}\|_p\leqq$$
$$\leqq\|(I+U_a^*T_a)|\mathfrak{D}_{T_a}\|_p+|\lambda|\|D_{T_a^*}\|_{2p}\|D_{T_a}\|_{2p}(1-|\lambda|)^{-1}\leqq\|D_{T_a}^2\|_p(1-|\lambda|)^{-1}.$$

(We have used that $\|(I+U_a^*T_a)|\mathfrak{D}_{T_a}\|_p=\left(\sum_n(1-(1-\mu_n)^{\frac{1}{2}})^p\right)^{\frac{1}{p}}\leqq\|D_{T_a}^2\|_p$ and that $\|D_{T_a}\|_{2p}=\|D_{T_a^*}\|_{2p}=(\|D_{T_a}^2\|_p)^{\frac{1}{2}}$.)

There exist unitary operators $U_1\in\mathscr{L}(\mathfrak{D}_T,\mathfrak{D}_{T_a})$, $U_2\in\mathscr{L}(\mathfrak{D}_{T_a^*},\mathfrak{D}_{T^*})$ such that $\Theta_T\left(\frac{\lambda+a}{1+\bar{a}\lambda}\right)=U_2\Theta_{T_a}(\lambda)U_1$. Then for $\Theta_0(\lambda)=U_1^*U_a^*U_2^*\Theta_T(\lambda)$ we have $I-\Theta_0\left(\frac{\lambda+a}{1+\bar{a}\lambda}\right)=$

$=U_1^*(I-\Theta_a(\lambda))U_1$ and $\|I-\Theta_0(\lambda)\|_p \leqq \|D_{T_a}^2\|_p \left(1-\left|\frac{\lambda-a}{1-\bar{a}\lambda}\right|\right)^{-1} \leqq \|D_{T_a}^2\|_p \left(1-\frac{|\lambda|+|a|}{1+|a||\lambda|}\right)^{-1}$.
Since $\|D_{T_a}^2\|_p = \|S^* D_T^2 S\|_p \leqq \|S\|^2 \|D_T^2\|_p$ where $\|S\| = \|(1-|a|^2)^{\frac{1}{2}}(I-\bar{a}T)^{-1}\| \leqq \leqq \left(\frac{1+|a|}{1-|a|}\right)^{\frac{1}{2}}$ we infer that $\|I-\Theta_0(\lambda)\|_p \leqq \left(\frac{1+|a|}{1-|a|}\right)^2 \|D_T^2\|_p (1-|\lambda|)^{-1}$. By Remark 3.2 $a \in D \setminus \sigma(T)$ can be chosen arbitrary small, therefore

$$\|I-\Theta_0(\lambda)\|_p \leqq \|D_T^2\|_p (1-|\lambda|)^{-1}.$$

Because $\|I-U\Theta_0(\lambda)\|_p \leqq \|I-\Theta_0(\lambda)\|_p + \|I-U\|_p \|\Theta_0(\lambda)\| \leqq \|I-\Theta_0(\lambda)\|_p + \|I-U\|_p$ if $U \in I+\mathscr{C}_p(\mathfrak{D}_T)$ is a unitary operator, the Lemma follows.

The next converse of Theorem 3.4 is true.

Theorem 3.5. *Let $\Theta \in H^\infty(\mathscr{L}(\mathfrak{E}, \mathfrak{E}_*))$ be a purely contractive analytic function. Let us assume that there exists a $\lambda_0 \in D$ such that $\Theta(\lambda_0)$ is invertible and there exist a $\lambda_1 \in D$ and an unitary operator $U \in \mathscr{L}(\mathfrak{E}, \mathfrak{E}_*)$ such that $U^*\Theta(\lambda_1) \in I+\mathscr{C}_p(\mathfrak{E})$. Then $S(\Theta)$ is a p-weak contraction.*

Proof. The characteristic function of $S=S(\Theta)$, Θ_S coincides with Θ. (Cf. [3], VI. 3.) So there exists a unitary operator $U_1 \in \mathscr{L}(\mathfrak{D}_S, \mathfrak{D}_{S^*})$ such that $U_1^*\Theta_S(\lambda_1) \in I+\mathscr{C}_p(\mathfrak{D}_S)$. Let us denote $S_1=S_{\lambda_1}=(S-\lambda_1 I)(I-\bar{\lambda}_1 S)^{-1}$. Since the characteristic function of S_1, $\Theta_1(\lambda)$ coincides with $\Theta_S\left(\frac{\lambda+\lambda_1}{1+\bar{\lambda}_1\lambda}\right)$ (cf. [3], ch. VI), there exists a unitary operator $U_2 \in \mathscr{L}(\mathfrak{D}_{S_1}, \mathfrak{D}_{S_1^*})$ such that $U_2^*\Theta_1(0) = -U_2^* S_1|\mathfrak{D}_{S_1} \in I+\mathscr{C}_p(\mathfrak{D}_{S_1})$. Therefore $(I-S_1^*S_1)|\mathfrak{D}_{S_1} = I-(S_1^*(-U_2))(-U_2^*S_1)|\mathfrak{D}_{S_1} \in \mathscr{C}_p(\mathfrak{D}_{S_1})$ and so $I-S_1^*S_1 \in \mathscr{C}_p$. $\Theta_1\left(\frac{\lambda_0-\lambda_1}{1-\bar{\lambda}_1\lambda_0}\right)$ being invertible we see that $\sigma(S_1) \neq D^-$ (cf. [3], VI). Therefore S_1 is a p-weak contraction and by Theorem 3.3 so is $S=(S_1)_{-\lambda_1}$.

Remark 3.6. Regarding Remark 3.2 we see that Theorem 3.5 remains valid if instead of the existence of $\lambda_0 \in D$ such that $\Theta(\lambda_0)$ is invertible we assume that there exist $\lambda_0', \lambda_0'' \in D$ such that $\Theta(\lambda_0')$ and $\Theta^\sim(\lambda_0'')$ are injections. (Cf. [3], VI. 4.)

Corollary 3.7. *Let $\Theta \in H^\infty(\mathscr{L}(\mathfrak{E}))$ be a puraly contractive analytic function. Let us assume that there exists a $\lambda_1 \in D$ for which $\Theta(\lambda_1) \in I+\mathscr{C}_p(\mathfrak{E})$ and there exist $\lambda_0', \lambda_0'' \in D$ such that $\Theta(\lambda_0')$ and $\Theta^\sim(\lambda_0'')$ are injections. Then $\Theta(\lambda) \in I+\mathscr{C}_p(\mathfrak{E})$ for all $\lambda \in D$ and $\Theta(\lambda)$ is invertible except isolated points in D.*

Proof. By Remark 3.6 $S(\Theta)$ is a p-weak contraction. So by Theorem 3.4 there exists a unitary operator $U \in \mathscr{L}(\mathfrak{E})$ such that $U\Theta(\lambda) \in I+\mathscr{C}_p(\mathfrak{E})$ for all $\lambda \in D$. In particular $U\Theta(\lambda_1) \in I+\mathscr{C}_p(\mathfrak{E})$. Since we have also $\Theta(\lambda_1) \in I+\mathscr{C}_p(\mathfrak{E})$ we infer that $U\in I+\mathscr{C}_p(\mathfrak{E})$ and so that $\Theta(\lambda)\in I+\mathscr{C}_p(\mathfrak{E})$ for all $\lambda\in D$. The other part of the theorem follows by Remark 3.2.

Corollary 3.8. *Let $\{m_n\}$ be an arbitrary sequence of non-constant inner functions.*

(i) *If there exists a $\lambda_1\in D$ such that $\sum_n |1-m_n(\lambda_1)|^p<\infty$, then for all $\lambda\in D$ we have $\sum_n |1-m_n(\lambda)|^p<\infty$.*

(ii) *If there exists a $\lambda_1\in D$ such that $\sum_n (1-|m_n(\lambda_1)|)^p<\infty$, then for all $\lambda\in D$ we have $\sum_n (1-|m_n(\lambda)|)^p<\infty$.*

Proof. (i) is an immediate consequence of the Corollary 3.7.

If $S=\bigoplus_n S(m_n)$ then $I-SS^*=\sum_n (1-|m_n(0)|^2)\langle\cdot,\varphi_n\rangle\varphi_n$ where $\{\varphi_n\}$ is an orthonormal system. So (ii) follows by Theorem 3.3 (i).

Now we regard some important corollaries of Theorem 3.4.

Corollary 3.9. *Let $p\geqq 1$ be an integer. If Θ is the contractive analytic function occurring in Theorem 3.4, then for all $\lambda\in D$ we have*

(i) $\|\Theta(\lambda)^{\overset{(p)}{\mathrm{Ad}}}\|\leqq D_p\exp\left[C_p(\|D_T^2\|_p(1-|\lambda|)^{-1}+\|I-U\|_p)^{\gamma_p}\right]$;

(ii) $|\overset{(p)}{\det}\,\Theta(\lambda)|\leqq\exp\left[C_p^*(\|D_T^2\|_p(1-|\lambda|)^{-1}+\|I-U\|_p)^p\right]$,

where C_p, D_p, γ_p are the constants from Theorem 2.10 and C_p^* is the constant from Lemma 2.3.

Proof. (i) is an immediate consequence of Theorem 3.4 and Theorem 2.10.

Let $A=I-X$ where $X\in\mathscr{C}_p$. Let us denote by $\{\lambda_n\}$, $\{s_n\}$ the characteristic values with algebraic multiplicities of X and $|X|$ respectively. Then by Lemma 2.3 and [2], ch. II. 3.1 we infer

$$|\overset{(p)}{\det}A| = \left|\prod_n\left[(1-\lambda_n)\exp\left(\lambda_n+\frac{1}{2}\lambda_n^2+\ldots+\frac{1}{p-1}\lambda_n^{p-1}\right)\right]\right|\leqq$$
$$\leqq\exp\left[C_p^*\sum_n|\lambda_n|^p\right]\leqq\exp\left[C_p^*\sum_n s_n^p\right]=\exp\left[C_p^*\|I-A\|_p^p\right].$$

(ii) follows from this relation and Theorem 3.4.

Theorem 3.10. *If T is a p-weak contraction ($p\geqq 1$ integer) and Θ is a contractive analytic function coinciding with the characteristic function of T such that $\Theta(\lambda)\in I+\mathscr{C}_p$ for all $\lambda\in D$, then $\Theta^{\overset{(p)}{\mathrm{Ad}}}$ and $\overset{(p)}{\det}\,\Theta$ are analytic functions on D.*

Proof. Let $\{P_n\}$ be a sequence of orthogonal projections of finite rank which converges strongly to the identity operator. Then $\Theta_n=P_n\Theta P_n+(I-P_n)\,\overset{(p)}{\det}(P_n\Theta P_n)$

is a contractive analytic function for every n, and $\{\|\Theta(\lambda)-\Theta_n(\lambda)\|_p\}_n$ converges to 0 for all $\lambda\in D$. Therefore $\{\overset{(p)}{\det}\,\Theta_n\}_n$ converges to $\overset{(p)}{\det}\,\Theta$, and by Theorem 3.4 and Corollary 3.9 this sequence is uniformly bounded in every compact subset of D. Now analycity of $\overset{(p)}{\det}\,\Theta$ follows by Vitali's theorem.

Regarding Theorem 2.10 (ii) analycity of $\overset{(p)}{\Theta}{}^{\mathrm{Ad}}$ follows similarly.

Definition 3.11. The $\mathscr{L}(\mathfrak{E}, \mathfrak{E}_*)$-valued analytiy function Θ is said to have the *general scalar multiple* $\delta(\lambda)$, if $\delta(\lambda)$ is a scalar valued analytic function, $\delta(\lambda)\not\equiv 0$, and there exists an $\mathscr{L}(\mathfrak{E}_*, \mathfrak{E})$-valued analytic function Ω such that for all $\lambda\in D$ we have

$$\Omega(\lambda)\Theta(\lambda)=\delta(\lambda)I_{\mathfrak{E}}, \quad \Theta(\lambda)\Omega(\lambda)=\delta(\lambda)I_{\mathfrak{E}_*}.$$

Theorem 3.12. *If T is a p-weak contraction ($p\geqq 1$ is real), then its characteristic function Θ_T has a general scalar multiple. Particularly $\overset{(q)}{\det}\,\Theta$ will be a general scalar multiple, where $q\geqq p$ is an arbitrary integer and Θ is a function coinciding with Θ_T and such that $\Theta(\lambda)\in I+\mathscr{C}_q$ for all $\lambda\in D$.*

Proof. Theorem follows by Theorems 3.4, 3.10 and 2.10 (i).

Remark 3.13. If $p>1$ real and T is a p-weak contraction, then Θ_T does not have generally a scalar multiple. Even it may happen that there is not a general scalar multiple belonging to some Hardy-class H^q ($q>0$). For example let $\{a_n\}$ be a sequence of complex numbers such that $0<|a_n|<1$, $\sum\limits_n (1-|a_n|)^p<\infty$ for all $p>1$, and $\sum\limits_n (1-|a_n|)=\infty$. Let us denote

$$m_n(\lambda)=\frac{|a_n|}{a_n}\frac{a_n-\lambda}{1-\bar{a}_n\lambda} \quad \text{and} \quad T=\bigoplus_n S(m_n).$$

Then T is a p-weak contraction for all $p>1$, but Θ_T does not have a general scalar multiple belonging to some class H^q. (Cf. [6] Theorem 2.3 and [3], ch. VI.)

4. The spectrum of a p-weak contraction

Let T be a p-weak contraction ($p\geqq 1$ integer). By Theorem 3.4 there exists an analytic function Θ coinciding with Θ_T such that $\Theta(\lambda)\in I+\mathscr{C}_p$ for all $\lambda\in D$. We can define $\overset{(p)}{\det}\,\Theta$ which will be an analytic function on D. Because $\overset{(p)}{\det}\,\Theta(\lambda)=0$ if and only if $\Theta(\lambda)$ is not invertible, we infer that $\sigma(T)\cap D$ coincides with the set of zeros of $\overset{(p)}{\det}\,\Theta$. (Cf. [3], VI. 4.) (For the unitary part T_u of T we have

$\sigma(T_u)\cap D=\emptyset$.) We can estimate the growth of $|\overset{(p)}{\det}\,\Theta(\lambda)|$ by Corollary 3.9. On the other hand there is a connection between the growth of absolute value and distribution of zeros of a function analytic on D. So we can get information about the distribution of points of $\sigma(T)\cap D$.

Definition 4.1. If $\alpha=\{\alpha_n\}$ is a sequence in D then $\tau_\alpha:=\inf\left\{x>0\,\middle|\,\sum_n(1-|\alpha_n|)^x<\infty\right\}$. If $\beta=\{\beta_n\}$ is an arbitrary sequence of non-zero numbers in $\mathbf{C}$ then

$$\tau'_\beta:=\inf\left\{x>0\,\middle|\,\sum_n\left(\frac{1}{|\beta_n|}\right)^x<\infty\right\}.$$

Definition 4.2. If f is an analytic function on D, then ϱ_f denotes the infimum of positive μ's which satisfy that there exists an $r_\mu<1$ such that for all $r_\mu<r<1$ we have $M_f(r)=\max\left\{|f(\lambda)|\,\middle|\,|\lambda|=r\right\}\leqq\exp[(1-r)^{-\mu}]$. We call ϱ_f the *order of the function f.*

Lemma 4.3. *If f is an analytic function on D and $\alpha=\{\alpha_n\}$ is the sequence of its zeros, taking them with multiplicities, then*

$$\tau_\alpha\leqq\varrho_f+1.$$

Proof. Let us denote $\beta=\{\beta_n=(1-|\alpha_n|)^{-1}\}$. For $\tau_\alpha=\tau'_\beta$ and $\tau'_\beta=\varlimsup_{r\to\infty}\dfrac{\ln\nu_\beta(r)}{\ln r}$, where $\nu_\beta(r)$ denotes the number of β_n's having absolute value less than r, (cf. [7], V. § 15), we have

$$\tau_\alpha=\varlimsup_{r\to1-0}\frac{\ln\nu_\beta((1-r)^{-1})}{-\ln(1-r)}=\varlimsup_{r\to1-0}\frac{\ln\nu_\alpha(r)}{-\ln(1-r)}=\varlimsup_{r\to1-0}\frac{\ln\nu_\alpha(1-e(1-r))}{-1-\ln(1-r)}$$

If $f(0)\neq0$ and $0<r<1$, then

$$\int_0^r\frac{\nu_\alpha(t)}{t}\,dt=\int_1^{(1-r)^{-1}}\frac{\nu_\beta(u)\,du}{u(u-1)}\geqq\frac{1-r}{r}\int_{(e(1-r))^{-1}}^{e(e(1-r)^{-1}}\frac{\nu_\beta(u)\,du}{u}\geqq$$

$$\geqq\frac{1-r}{r}\nu_\beta((e(1-r))^{-1})=\frac{1-r}{r}\nu_\alpha(1-e(1-r)).$$

Moreover if $|f(0)|=1$, then $\displaystyle\int_0^r\frac{\nu_\alpha(t)\,dt}{t}\leqq\ln M_f(r)$ (cf. [7], V. § 15), so in this case $\nu_\alpha(1-e(1-r))\leqq\dfrac{r}{1-r}\ln M_f(r)$. We infer that if f is an arbitrary function having 0 as a zero with multiplicity n, and $\dfrac{3}{4}<r<1$ then

$$\nu_\alpha(1-e(1-r))\leqq\frac{r}{1-r}\ln M_f(r)+\frac{r}{1-r}\ln\frac{n!}{2^{-n}|f^{(n)}(0)|}+n.$$

Therefore

$$\tau_\alpha \leqq 1+\varlimsup_{r\to 1-0}\frac{\ln\ln M_f(r)}{-\ln(1-r)}.$$

It is easily seen that

$$\varrho_f = \varlimsup_{r\to 1-0}\frac{\ln\ln M_f(r)}{-\ln(1-r)},$$

so the Lemma is proved.

Theorem 4.4. *If T is a p-weak contraction ($p\geqq 1$ integer) and $\sigma(T)\cap D= =\{\lambda_n\}=\lambda$, then $\tau_\lambda\leqq p+1$.*

Proof. The Theorem follows by Corollary 3.9 and Lemma 4.3.

This estimation is not exact in the case $p=1$. Indeed then $\det\Theta\in H^\infty$ and so $\sum_n(1-|\lambda_n|)<\infty$. There is a question whether it is exact in the case $p\geqq 2$. In Theorem 3.12 we verified the existence of a general scalar multiple of order p of Θ_T if T is a p-weak contraction. It remains a question whether there is a general scalar multiple of order $(p-1)$ of Θ_T for arbitrary p.

We give exact estimation for a special class of operators.

Lemma 4.5. *Let $\{m_i\}_i$ be a sequence of inner functions. Let us denote $\{\alpha_n^{(i)}\}_n$ the zeros of m_i with multiplicities. If there exists a $\lambda_1\in D$ such that $\sum_i(1-|m_i(\lambda_1)|)^p<\infty$ ($p\geqq 1$ real) then $\sum_i\sum_n(1-|\alpha_n^{(i)}|)^p<\infty$.*

Proof. By Corollary 3.8 we know $\sum_i(1-|m_i(0)|)^p<\infty$. Regarding the factorization of m_i into the product of a Blaschke product and an inner function non-vanishing on D, we see that $\sum_i(1-\prod_n|\alpha_n^{(i)}|)^p\leqq\sum_i(1-|m_i(0)|)^p<\infty$. There exists a $\delta>0$ such that for every $\frac{1}{2}\leqq x\leqq 1$ we have $1-x\leqq -\ln x\leqq\delta(1-x)$. We may assume that $\frac{1}{2}\leqq\prod_n|\alpha_n^{(i)}|\leqq 1$ for every i. So for arbitrary i

$$0\leqq\sum_n(1-|\alpha_n^{(i)}|)\leqq -\sum_n\ln|\alpha_n^{(i)}| = -\ln\Big(\prod_n|\alpha_n^{(i)}|\Big)\leqq\delta\Big(1-\prod_n|\alpha_n^{(i)}|\Big).$$

Therefore $\sum_{i,n}(1-|\alpha_n^{(i)}|)^p\leqq\sum_i\Big(\sum_n(1-|\alpha_n^{(i)}|)\Big)^p\leqq\delta^p\sum_i\Big(1-\prod_n|\alpha_n^{(i)}|\Big)^p<\infty$. The Lemma is proved.

Theorem 4.6. *Let $\{m_i\}_i$ be an arbitrary sequence of inner functions, and $S=\bigoplus_i S(m_i)$. If S is a p-weak contraction ($p\geqq 1$ real) and $\sigma(S)\cap D=\{\lambda_n\}_n$, then $\sum_n(1-|\lambda_n|)^p<\infty$. (It is easily seen that this is an exact estimation.)*

Proof. Since S is a p-weak contraction, we have $\sum_i(1-|m_i(0)|)^p<\infty$. By Lemma 4.5 we infer $\sum_{i,n}(1-|\alpha_n^{(i)}|)^p<\infty$, where $\{\alpha_n^{(i)}\}_n$ is the sequence of zeros of

m_i without multiplicities. For Θ_S coincides with $\begin{pmatrix} m_1 & & & 0 \\ & m_2 & & \\ & & m_3 & \\ 0 & & & \ddots \end{pmatrix}$, so $\sigma_p(S)\cap D=$ $=\overline{\sigma_p(S^*)}\cap D=\{\alpha_n^{(i)}\}_{n,i}\neq D$. (Cf. [3], VI. 4.) Then by Remark 3.2 we see that $\sigma(S)\cap$ $\cap D=\sigma_p(S)\cap D$, therefore $\sigma(S)\cap D=\{\alpha_n^{(i)}\}_{n,i}$ and the Theorem follows.

5. The spectra of contractions of class $C_{\cdot 1}$

Lemma 5.1. *If $\Theta\in H^\infty(\mathscr{L}(\mathfrak{E},\mathfrak{E}_*))$ is an outer function, then for all $0<r<1$ $\Theta_r(\lambda)=\Theta(r\lambda)$ will be also an outer function.*

Proof. Let us denote $U_r\in\mathscr{L}(H^2(\mathfrak{E}))$ the operator defined by $(U_r u)(\lambda)=u(r\lambda)$, $u\in H^2(\mathfrak{E})$. Regarding the decomposition $H^2(\mathfrak{E})=\bigoplus_{n=0}^{\infty} U_+^n\mathfrak{E}$, where U_+ is the operator of multiplication by λ, U_r has the form $U_r=\bigoplus_{n=0}^{\infty} r^n I_{U_+^n\mathfrak{E}}$. So U_r is a quasiaffinity. Let $U_{*r}\in\mathscr{L}(H^2(\mathfrak{E}_*))$ be defined similarly. We can easily see that $\Theta_r U_r=U_{*r}\Theta$. Therefore $(\Theta_r H^2(\mathfrak{E}))^-=(\Theta_r U_r H^2(\mathfrak{E}))^-=(U_{*r}\Theta H^2(\mathfrak{E}))^-=(\Theta H^2(\mathfrak{E}))^-=$ $=H^2(\mathfrak{E}_*)$. That is Θ_r is outer and the Lemma is proved.

Remark 5.2. The converse of Lemma 5.1 is not true. Namely, there exists a 2-weak contraction T of class C_{01} (cf. [8]). (For the definition of the class C_{01} see [3], II. 4.) Θ_T is an outer function (cf. [3], VI. 3) which by Theorem 3.12 has a general scalar multiple. Then by Lemma 5.1, for all $0<r<1$ $\Theta_T(r\lambda)$ will be outer, and it is easily seen that it has a scalar multiple. So $\Theta_T^{\sim}(r\lambda)$ is also an outer function for all $0<r<1$ (cf. [3], V. 6), but $\Theta_T^{\sim}(\lambda)$ is not outer for $T\in C_{0\cdot}$.

From the above Lemma, using Theorem 6.2 of [3] ch. V, we infer.

Theorem 5.3. *If the outer function Θ has a general scalar multiple then $\Theta(\lambda)$ is boundedly invertible for all $\lambda\in D$.*

By this Theorem we get the next:

Corollary 5.4. *Let T be a c.n.u. contraction of class $C_{\cdot 1}$ whose characteristic function admits a general scalar multiple. Particularly this is the case if T is a c.n.u. p-weak contraction of class $C_{\cdot 1}$. Then the spectrum $\sigma(T)$ is situated on the circle C.*

Remark 5.5. It would be interesting to know whether Proposition 4.4 of [3], ch. VI remains true replacing scalar multiple by general scalar multiple. The answer depends on the validity of Proposition 6.7 of [3], ch. V in this general situation.

6. p-weak contractions of class C_0

The next theorem is a generalization of [1], Proposition 4.3.

Theorem 6.1. *Let T be a C_0 operator and let S be its Jordan-model. If S is a p-weak contraction ($p \geqq 1$ real), then T is also a p-weak contraction.*

Proof. Regarding the proof of [1], Proposition 4.3 Theorem follows from the next Lemma.

Lemma 6.2. *Let $\{a_k\}_k$, $\{b_k\}_k$ be increasing sequences such that $0<a_k\leqq 1$, $0<b_k\leqq 1$ for every k and $\prod_{k=1}^{n} a_k \leqq \prod_{k=1}^{n} b_k$ for every n. If $\sum_{k=1}^{\infty}(1-a_k)^p<\infty$, where $p\geqq 1$ real, then $\sum_{k=1}^{\infty}(1-b_k)^p<\infty$.*

Proof. From the assumption it follows that $\sum_{k=1}^{n} \log\frac{1}{a_k} \geqq \sum_{k=1}^{n} \log\frac{1}{b_k}$ for arbitrary n. Since $\left\{\log\frac{1}{a_k}\right\}_k$ and $\left\{\log\frac{1}{b_k}\right\}_k$ are decreasing sequences we infer by Lemma 3.4 of [2], ch. II that $\sum_{k=1}^{n}\frac{1}{a_k} \geqq \sum_{k=1}^{n}\frac{1}{b_k}$, that is $\sum_{k=1}^{n}\frac{1-a_k}{a_k} \geqq \sum_{k=1}^{n}\frac{1-b_k}{b_k}$ for every n. $\left\{\frac{1-a_k}{a_k}\right\}_k$ and $\left\{\frac{1-b_k}{b_k}\right\}_k$ being also decreasing sequences we can employ again the above Lemma. So we get $\sum_{k=1}^{\infty}\left(\frac{1-a_k}{a_k}\right)^p \geqq \sum_{k=1}^{\infty}\left(\frac{1-b_k}{b_k}\right)^p$. It follows from the assumption $\sum_{k=1}^{\infty}(1-a_k)^p<\infty$ that $\lim_{k\to\infty} a_k=1$, so $\sum_{k=1}^{\infty}\left(\frac{1-a_k}{a_k}\right)^p<\infty$. Therefore $\sum_{k=1}^{\infty}(1-b_k)^p \leqq \sum_{k=1}^{\infty}\left(\frac{1-b_k}{b_k}\right)^p \leqq \sum_{k=1}^{\infty}\left(\frac{1-a_k}{a_k}\right)^p<\infty$. The Lemma is proved.

Remark 6.3. With a slight modification of the example of [1], Remark 4.4 we can show that the converse of Theorem 6.1 is in general false. Namely let μ be a finite non-negative measure on $[0, 2\pi]$, singular with respect to Lebesgue measure and without atoms. Let us assume that $\mu([0, 2\pi])=1$. For every n there exists a decomposition of $[0, 2\pi]$ into disjoint intervals $i_1^{(n)}, \dots, i_{2^n}^{(n)}$ such that $\mu(i_k^{(n)})=2^{-n}$ for $k=1, \dots, 2^n$. Let $m_{k,n}(\lambda)$ and $m(\lambda)$ be inner functions defined by

$$m_{k,n}(\lambda) = \exp\left[-\int_{i_k^{(n)}} \frac{e^{it}+\lambda}{e^{it}-\lambda}\, d\mu(t)\right]$$

($n=1, 2, \dots;\ k=1, \dots, 2^n$), and

$$m(\lambda) = \exp\left[-\int_0^{2\pi} \frac{e^{it}+\lambda}{e^{it}-\lambda}\, d\mu(t)\right].$$

Let T, S be the operators $T=\bigoplus_{n=1}^{\infty}\left(\bigoplus_{k=1}^{2^n} S(m_{k,n})\right)$ and $S=S(m)\oplus S(m)\oplus\dots$. Then S is the Jordan-model of T (cf. [9]). Since $\sum_{n=1}^{\infty}\sum_{k=1}^{2^n}(1-|m_{k,n}(0)|^2)^p=$ $=\sum_{n=1}^{\infty}2^n\left(1-\exp\left(-\frac{2}{2^n}\right)\right)^p\leqq 2^p\sum_{n=1}^{\infty}\left(\frac{2}{2^p}\right)^n<\infty$ if $p>1$, it follows that T is a p-weak contraction for all real number $p>1$. On the other hand, $I-SS^*$ is not compact.

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Some general fixed point theorems

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In the late 1960's and early 1970's a large number of fixed point papers were written involving definitions which are generalizations of the original contractive definition attributed to Banach. A classification and comparison of many of these definitions appears in [17]. More recently, several authors have made improvements by recognizing that the contractive definitions need not hold for all points in the space. For example, B. FISHER [7] has proved several fixed point theorems involving contractive definitions which are satisfied only for points x, fy, for x, y in the space. (See also [18].) ĆIRIĆ [3] made the observation that certain contractive definitions imply the boundedness of $O(x)$, the orbit of x, for each x in the space, where $O(x)=\{x, f(x), f^2(x), \ldots\}$. This idea has been utilized by HEGEDŰS [11]. Other authors have used a contractive definition involving a function $\varphi\colon R_+\to R_+$, which is nondecreasing and satisfies $\varphi(t)<t$ for each $t>0$, where $R_+=[0, \infty)$. (See, e.g. [1].)

In this paper we establish several fixed point theorems involving hypotheses weak enough to include a number of fixed point theorems as special cases.

Our first result is the following, which is a generalization of Theorem 4.1 of the first author [16].

Let X be a topological space. A function $G\colon X\to R_+$ is called f-orbitally lower semicontinuous at a point $p\in X$ if, for every $x_0\in X$, $x_{n_k}\to p$ implies $G(p)\leqq \liminf_k G(x_{n_k})$, where $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$, and $\{x_n\}$ is defined by $x_{n+1}=f(x_n)$; i.e., $\{x_n\}=0(x_0)$.

Theorem 1. *Let f be a selfmap of a topological space X, and d a non-negative, real valued function defined on $X\times X$ such that $d(x, y)=d(y, x)$ and $d(x, y)=0$ iff $x=y$. If there exists a point $u\in X$ such that $\lim_n d(f^{n+1}(u), f^n(u))=0$, and, if $\{f^n(u)\}$ has a convergent subsequence with limit $p\in X$, then p is a fixed point of f if and only if $G(x)=d(x, f(x))$ is f-orbitally lower semicontinuous at p.*

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Proof. Suppose $\{f^{n_k}(u)\}$ converges to a fixed point p of f. Then $0=G(p)\leqq \leqq \liminf_k G(f^{n_k}(u))$.

Conversely, if G is f-orbitally lower semicontinuous at p, then

$$0=\lim_n d(f^{n+1}(u), f^n(u)) = \liminf_k d(f^{n_k+1}(u), f^{n_k}(u)) \geqq d(p, f(p)),$$

since, for each k sufficiently large, there exists an integer n_k satisfying $n_k\geqq k$.

Theorem 1 includes Theorem 2 of [15].

Corollary 1. *Let X be a metric space, $f: X\to X$, $\varphi: X\to R_+$ such that, there exists a point $u\in X$ with $d(x, f(x))\leqq\varphi(x)-\varphi(f(x))$ for each $x\in O(u)$, and $\bar{O}(u)$ is complete. Then*

(i) $\lim_n f^n(u)=p$ exists, and

(ii) p is a fixed point of f if and only if $G=d(x, f(x))$ is f-orbitally lower semicontinuous at p.

The proof of Corollary 1 follows from Theorem 1 and the following

Lemma (Siegel [20]. *Let $\{x_n\}$ be a sequence in $O(u)$ such that $d(x_n, x_{n+1})\leqq \leqq\varphi(x_n)-\varphi(x_{n+1})$ for all n, φ, u as in Corollary 1. Then $\lim_n x_n$ exists.*

Corollary 1 compares well with Caristi's Theorem [2] and Theorem 1 of [12] as special cases.

Let $\delta(O(x))$ denote the *diameter of the orbit of* x.

The following result is an extension of Hegedűs [11] and Theorem 1 of Daneš [4].

Theorem 2. *Let f be a selfmap of a metric space (X, d) satisfying:*

(i) $\delta(O(x))<\infty$ for each $x\in X$.

(ii) There exists $u\in X$ such that $O(u)$ has a cluster point $p\in X$.

(iii) There exists a map $\varphi: R_+\to R_+$ which is nondecreasing, continuous from the right and satisfies $\varphi(t)<t$ for each $t>0$ and the inequality,

$$d(f(x), f^2(y)) \leqq \varphi(\delta(O(x)\cup O(f(y)))) \quad \textit{for each} \quad x, y\in X.$$

Then p is the unique fixed point of f and $\lim_n f^n(u)=p$.

Proof. Define $\varrho_n=\delta(O(f^n(u)))$. From (i), ϱ_n is finite for each n. Since $\varrho_{n+1}\leqq\varrho_n$ for each n, $\{\varrho_n\}$ converges to some number $\varrho\geqq 0$.

For each $j>i\geqq n+1$, from (iii),

$$d(f^i(u), f^j(u)) \leqq \varphi(\delta(O(f^{i-1}(u))\cup O(f^{j-2}(u)))) \leqq \varphi(\delta(O(f^n(u)))) = \varphi(\varrho_n),$$

so that $\varrho_{n+1}\leqq\varphi(\varrho_n)$ for each n. Since φ is continuous from the right, $\varrho\leqq\varphi(\varrho)$, which implies $\varrho=0$. Therefore $\{f^n(u)\}$ is Cauchy, and $f^n(u)\to p$ by (ii).

For each $\varepsilon>0$ there exists an integer N such that $n\geqq N$ implies $d(f^n(u), p)<\varepsilon$.

For any integers $m>0$ and $n>N$, from (iii) it follows

$$d(p, f^m(p)) \leqq d(p, f^{n+1}(u)) + d(f(f^{m-1}(p)), f^2(f^{n-1}(u))) \leqq$$
$$\leqq d(p, f^{n+1}(u)) + \varphi(\delta(O(f^{m-1}(p)) \cup O(f^{n-1}(u)))) \leqq \varepsilon + \varphi(\max\{2\varepsilon, \delta(O(p))+\varepsilon\}).$$

From the Lemma of [11], $\delta(O(p)) = \sup_m d(p, f^m(p))$, so that we have

$$\delta(O(p)) \leqq \varepsilon + \varphi(\max\{2\varepsilon, \delta(O(p))+\varepsilon\}).$$

Since ε is arbitrary, $\delta(O(p)) \leqq \varphi(\delta(O(p)))$, so that $O(p)=0$, which implies $\delta(O(p))=0$. Therefore $p=f(p)$.

Uniqueness follows from (iii).

The next result is an extension of Theorem 2 to 2-metric spaces, and is a generalization of Theorem 1 of [19].

A 2-metric space is a space X in which, for each triple of points a, b, c, there exists a real-valued nonnegative function ϱ satisfying

(1a) for each pair of points $a, b, a \neq b$, of X, there exists a point $c \in X$ such that $\varrho(a, b, c) \neq 0$,

(1b) $\varrho(a, b, c)=0$ when at least two of the points are equal,

(2) $\varrho(a, b, c) = \varrho(a, c, b) = \varrho(b, c, a)$, and

(3) $\varrho(a, b, c) \leqq \varrho(a, b, d) + \varrho(a, d, c) + \varrho(d, b, c)$.

For other properties of 2-metric spaces the reader may consult [5], [6], [8]—[10], and [21]. Fixed point theorems for 2-metric spaces appear in [13], [14], and [19].

For a set $A \subset X$, define $\delta_a(A) = \sup\{\varrho(x, y, a) | x, y \in A\}$. In a 2-metric space a sequence $\{x_n\}$ is called bounded if, for each $a \in X$, $\sup_{m,n} \varrho(x_m, x_n, a) < \infty$, and Cauchy, if, for each $\varepsilon>0$ there exists an integer $N=N(a, \varepsilon)$ such that $\varrho(x_m, x_n, a) < \varepsilon$ for all $m, n > N$. ϱ is always continuous in one coordinate.

Theorem 3. *Let f be a selfmap of a 2-metric space X with the following properties:*

(i) *$\delta_a[O(x) \cup O(y)]$ is finite for each $x, y, a \in X$.*

(ii) *There exists $u \in X$ such that $O(u)$ has a cluster point $p \in X$.*

(iii) *There exists a map $\varphi: R_+ \to R_+$ which is semicontinuous from the right, nondecreasing, and satisfies $\varphi(t) < t$ for each $t > 0$.*

(iv) *f satisfies $\varrho(f(x), f^2(y), a) \leqq \varphi[\delta_a(O(x) \cup O(f(y)))]$ for each $x, y, a \in X$.*
Then p is the unique fixed point of f, and $\lim_n f^n(u) = p$.

Proof. Let n be an arbitrary integer, i, j integers satisfying $i > j \geqq n$.

$$\varrho(f^i(u), f^j(u), a) = \varrho(f(f^{j-1}(u), f^2(f^{i-2}(u)), a) \leqq$$
$$\leqq \varphi[(\delta_a O(f^{j-1}(u)) \cup O(f^{i-2}(u)))] \leqq \varphi[\delta_a(O(f^{j-1}(u)))] \leqq \delta_a(O(f^{j-1}(u))) < \infty$$

by (iv). Taking the supremum over all $i>j\geqq n$ we obtain

$$\delta_a[O(f^n(u))] \leqq \varphi(\delta_a[O(f^{n-1}(u))]) \leqq \delta_a[O(f^{n-1}(u))]. \tag{4}$$

If we define $\delta_n=\delta_a[O(f^n(u))]$, then $\{\delta_n\}$ is nonincreasing and hence converges to a real number $\delta\geqq 0$. Also, $\delta_{n+1}\leqq\varphi(\delta_n)$. From (iii) it follows that $\delta\leqq\varphi(\delta)$, and hence $\delta=0$.

For each $m>n$,

$$\varrho(f^m(u), f^n(u), a) \leqq \varphi(\delta_a[O(f^{n-1}(u))]) \leqq \delta_a[O(f^{n-1}(u))] = \delta_{n-1} \to 0 \quad \text{as} \quad n\to\infty.$$

Therefore $\{f^n(u)\}$ is Cauchy and, from (ii), converges to p.

It remains to show that p is a fixed point for f. As in the proof of (4) it can be shown that

$$\delta_a[O(f^n(u)\cup f^n(p))] \leqq \varphi(\delta_a[O(f^{n-1}(u)\cup f^{n-1}(p))]),$$

and hence, that

$$\lim_n \delta_a[O(f^n(u)\cup f^n(p))] = 0, \quad \text{for each} \quad a\in X. \tag{5}$$

Using (3),

$$\begin{aligned}\varrho(p, f^n(p), a) &\leqq \varrho(p, f^n(p), f^n(u))+\varrho(p, f^n(u), a)+\varrho(f^n(u), f^n(p), a) \leqq \\ &\leqq \delta_p[O(f^n(u)\cup f^n(p))]+\varrho(p, f^n(u), a)+\delta_a[O(f^n(u)\cup f^n(p))].\end{aligned}$$

Taking the limit as $n\to\infty$, and using (5), we have

$$\lim_n \varrho(p, f^n(p), a) = 0. \tag{6}$$

Now let $\delta_n=\delta_a[O(f^n(p))]$. Again using (3), for any $n>m>0$,

$$\begin{aligned}\varrho(p, f^m(p), a) &\leqq \varrho(p, f^m(p), f^n(p))+\varrho(p, f^n(p), a)+\varrho(f^n(p), f^m(p), a)\\ &\leqq \varrho(p, f^m(p), f^n(p))+\varrho(p, f^n(p), a)+\delta_1.\end{aligned}$$

Taking the limit as $n\to\infty$, and using (5) and (6), one obtains

$$\varrho(p, f^m(p), a) \leqq \delta_1. \tag{7}$$

If, for any $n, \delta_n\neq 0$, then, from (4) and (iii), $\delta_{n+1}\leqq\varphi(\delta_n)<\delta_n$. Also,

$$\delta_n = \max\{\sup_{m>n} \varrho(f^n(p), f^m(p), a), \sup_{m,j>n} \varrho(f^m(p), f^j(p), a)\}.$$

If $\delta_n>0$, then $\sup_{m,j>n} \varrho(f^m(p), f^j(p), a)\leqq\varphi(\delta_n)<\delta_n$, so that

$$\delta_n = \sup_{m>n} \varrho(f^n(p), f^m(p), a). \tag{8}$$

If $\delta_0\neq 0$, then, taking the supremum of (7) for $m>0$, and using (8), yields $\delta_0\leqq\delta_1$. But $\delta_1\leqq\varphi(\delta_0)<\delta_0$, so that $\delta_0<\delta_0$, a contradiction.

Therefore $\delta_0=0$ and p is a fixed point for f.

To establish uniqueness, suppose w is also a fixed point of f. From (iv),

$$\varrho(p, w, a) = \varrho(f(p), f^2(w), a) \leqq \varphi(\delta_a[O(p) \cup O(f(w))]) =$$

$$= \varphi(\delta_a[O(p) \cup O(w)]) = \varphi(\varrho(p, w, a)).$$

From the definition of φ, $\varrho(p, w, a) \neq 0$ yields the contradiction $\varrho(p, w, a) < \varrho(p, w, a)$. Therefore $\varrho(p, w, a)=0$ for all $a \in X$, i.e., $p=w$.

Remark. Wong [22] has noted that, for nondecreasing functions $\varphi: R_+ \to R_+$, φ is continuous from the right if and only if φ is upper semicontinuous from the right. It is for this reason that the theorems of this paper have been phrased in terms of φ being continuous from the right.

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Uniform lattices

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In this paper we shall give a method of embedding a lattice into a uniform lattice. We shall use the notation and the terminology of [1] and [2]. Let us recall some of this terminology first.

If $L, \wedge, \vee$ is any lattice, and $e \in L$, then we denote the principal ideal generated by e in L by eL. If e and f are any elements of L such that $\alpha: eL \to fL$ is an isomorphism of eL onto fL, then we shall call α a *partial isomorphism* of L. The set of partial isomorphisms of L forms an inverse subsemigroup T_L of the inverse semigroup $\mathscr{I}_L$ of one-to-one partial transformations of L; T_L will be called the *Munn semigroup* of L [3]. We define an equivalence relation $\mathscr{U}_L$ on L by

$$\mathscr{U}_L = \{(e, f) \in L \times L \mid eL \cong fL\}.$$

The lattice L will be called *uniform* if $\mathscr{U}_L = L \times L$. It can be shown that L is uniform if and only if $L, \wedge$ is the semilattice of idempotents of some bisimple inverse semigroup [3].

If L is any lattice, then the automorphism group of L will be denoted by Aut (L), the endomorphism semigroup of L will be denoted by End (L), and the lattice of congruences of L will be denoted by $\theta(L)$.

We now proceed with our construction. Let $L, \wedge, \vee$ be a lattice. Let Z^+ denote the set of positive integers. For any $e \in L$ and any $i \in Z^+$ let $X_e^{(i)}$ be a set and

$$\varkappa_e^{(i)}: eL \to X_e^{(i)}$$

a one-to-one mapping of eL onto $X_e^{(i)}$. We shall thereby suppose that $X_f^{(j)} \cap X_e^{(i)} = \square$ if $i \neq j$ or $e \neq f$, and that $\left(\bigcup_{e \in L} \left(\bigcup_{i \in Z^+} X_e^{(i)}\right)\right) \cap L = \square$. Let us put $X_e = \bigcup_{i \in Z^+} X_e^{(i)}$ for all $e \in L$, and let $X = \bigcup_{e \in L} X_e$. If Y is a subset of X, then we shall put

$$Y_e = Y \cap X_e, \quad Y_e^{(i)} = Y \cap X_e^{(i)}$$

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for all $e\in L$ and all $i\in Z^+$. Let $\mathscr{A}$ be a set, the elements of which are the subsets Y of X which satisfy the following conditions:

(i) there is only a finite number of pairs $(e, i)\in L\times Z^+$ for which $Y_e^{(i)}\neq X_e^{(i)}$,

(ii) for every $(e, i)\in L\times Z^+$, either $Y_e^{(i)}=\Box$ or $Y_e^{(i)}\varkappa_e^{(i)-1}$ is of the form gL for some $g\in eL$.

Remark that $X\in\mathscr{A}$. Clearly $\mathscr{A}$ is a subset of the power set $P(X)$.

Let $\mathscr{B}$ be the subset of $P(X\cup L)$ which is defined by

$$\mathscr{B} = \{eL\cup Y|e\in L,\ Y\in\mathscr{A}\}\cup\mathscr{A}.$$

$\mathscr{B}, \subseteqq$ is a partially ordered set. It is easy to check that $\mathscr{B}, \subseteqq$ is in fact a lattice. Let us for instance compute the l.u.b. and the g.l.b. of $eL\cup V$ and $fL\cup W$, $e, f\in L$, $V, W\in\mathscr{A}$, in $\mathscr{B}$. It is obvious that

$$\text{g.l.b.}\,(eL\cup V,\ fL\cup W) = (eL\cup V)\cap(fL\cup W) = (e\wedge f)L\cup(V\cap W)$$

since $\mathscr{B}$ is closed for taking intersections. Let us now define an element U of $\mathscr{A}$ in the following way: for every $(e, i)\in L\times Z^+$ we take

$$\begin{aligned} U_e^{(i)} &= \Box & &\text{if } V_e^{(i)} = W_e^{(i)} = \Box,\\ U_e^{(i)} &= W_e^{(i)} & &\text{if } V_e^{(i)} = \Box,\\ U_e^{(i)} &= V_e^{(i)} & &\text{if } W_e^{(i)} = \Box, \end{aligned}$$

and in case $V_e^{(i)}=(vL)\varkappa_e^{(i)}$, $W_e^{(i)}=(wL)\varkappa_e^{(i)}$, take

$$U_e^{(i)} = \big((v\vee w)L\big)\varkappa_e^{(i)}.$$

Then

$$\text{l.u.b.}\,(eL\cup V,\ fL\cup W) = (e\vee f)L\cup U.$$

From this it follows that the mapping

$$\varphi\colon L\to\mathscr{B},\quad e\to eL\cup X$$

embeds L isomorphically as a dual ideal in $\mathscr{B}$. It is therefore possible to conceive a lattice L_1 which contains L as a dual ideal, and an isomorphism $\varphi_1\colon L_1\to\mathscr{B}$ of L_1 onto $\mathscr{B}$ which extends the isomorphism φ of L into $\mathscr{B}$. We shall investigate the embedding of L into L_1 in several lemmas.

Lemma 1. $L\times L\subseteqq\mathscr{U}_{L_1}$.

Proof. Let us consider any element e of L. Any element in the principal ideal of $eL\cup X$ in $\mathscr{B}$ is of the form $gL\cup Y$ or of the form Y, where $g\in eL$ and $Y\in\mathscr{A}$. Let φ_e be the mapping of the principal ideal of $eL\cup X$ in $\mathscr{B}$ onto the principal ideal of X in $\mathscr{B}$ which is defined by

$$(gL\cup Y)\varphi_e = (gL)\varkappa_e^{(1)}\cup\Big(\bigcup_{i\in Z^+} Y_e^{(i)}\varkappa_e^{(i)-1}\varkappa_e^{(i+1)}\Big)\cup(Y\setminus Y_e)$$

and

$$Y\varphi_e = \big(\bigcup_{i\in Z^+} Y_e^{(i)}\varkappa_e^{(i)-1}\varkappa_e^{(i+1)}\big)\cup(Y\setminus Y_e).$$

It is easy to verify that φ_e is a partial isomorphism of $\mathscr{B}$. Thus $(X, eL\cup X)\in\mathscr{U}_{\mathscr{B}}$ for all $e\in L$. From this it follows that $(eL\cup X, fL\cup X)\in\mathscr{U}_{\mathscr{B}}$ for all $e, f\in L$. Hence, $(e,f)\in\mathscr{U}_{L_1}$, for all $e,f\in L$, and so $L\times L\subseteqq\mathscr{U}_{L_1}$.

Lemma 2. *Every partial isomorphism α of L can be extended to a partial isomorphism $\alpha^{(1)}$ of L_1 in such a way that the mapping*

$$\psi_1\colon T_L\to T_{L_1},\quad \alpha\to\alpha^{(1)}$$

is an isomorphism of T_L into T_{L_1}.

Proof. Let $\alpha\colon eL\to fL$ be any partial isomorphism of L, and let us define the partial isomorphism $\bar\alpha$ of the principal ideal of $eL\cup X$ in $\mathscr{B}$ onto the principal ideal of $fL\cup X$ in $\mathscr{B}$ by

$$(gL\cup Y)\bar\alpha = (g\alpha)L\cup Y,\quad Y\bar\alpha = Y,\quad g\in eL,\quad Y\in\mathscr{A}.$$

Let $\alpha^{(1)}=\varphi_1\bar\alpha\varphi_1^{-1}$. Clearly $\alpha^{(1)}$ is a partial isomorphism of L_1 which maps eL_1 isomorphically onto fL_1, and the restriction of $\alpha^{(1)}$ to L is precisely α. Let us now consider the mapping $\psi_1\colon T_L\to T_{L_1}, \alpha\to\alpha^{(1)}$. We have

$$(\alpha\beta)\psi_1 = \varphi_1\overline{\alpha\beta}\varphi_1^{-1} = \varphi_1\bar\alpha\bar\beta\varphi_1^{-1} = (\varphi_1\bar\alpha\varphi_1^{-1})(\varphi_1\bar\beta\varphi_1^{-1}) = (\alpha\psi_1)(\beta\psi_1).$$

Since ψ_1 is clearly injective it follows that ψ_1 is an isomorphism of T_L into T_{L_1}.

Lemma 3. *Every endomorphism γ of L can be extended to an endomorphism $\gamma^{(1)}$ of L_1, in such a way that the mapping*

$$\xi_1\colon \mathrm{End}(L)\to\mathrm{End}(L_1),\quad \gamma\to\gamma^{(1)}$$

is an isomorphism of End (L) *into* End (L_1).

Proof. Let γ be any element of End (L), and let us define the endomorphism $\bar\gamma$ of $\mathscr{B}$ by

$$(eL\cup Y)\bar\gamma = (e\gamma)L\cup Y,\ Y\bar\gamma = Y,\quad e\in L,\ Y\in\mathscr{A}.$$

Let $\gamma^{(1)}=\varphi_1\bar\gamma\varphi_1^{-1}$. Then $\gamma^{(1)}\in\mathrm{End}(L_1)$, and the restriction of $\gamma^{(1)}$ to L is precisely γ. The mapping $\xi_1\colon\mathrm{End}(L)\to\mathrm{End}(L_1), \gamma\to\gamma^{(1)}$ is clearly injective, and for every $\gamma,\delta\in\mathrm{End}(L)$ we have

$$(\gamma\delta)\xi_1 = \varphi_1\overline{\gamma\delta}\varphi_1^{-1} = \varphi_1\bar\gamma\bar\delta\varphi_1^{-2} = (\varphi_1\bar\gamma\varphi_1^{-1})(\varphi_1\bar\delta\varphi_1^{-1}) = (\gamma\xi_1)(\delta\xi_1)$$

Thus ξ_1 is an isomorphism of End (L) into End (L_1).

Lemma 4. *Every automorphism* γ *of* L *can be extended to an automorphism* $\gamma^{(1)}$ *of* L_1, *and the mapping*

$$\xi_1|\mathrm{Aut}(L)\colon \mathrm{Aut}(L) \to \mathrm{Aut}(L_1), \quad \gamma \to \gamma^{(1)}$$

is an isomorphism of $\mathrm{Aut}(L)$ *into* $\mathrm{Aut}(L_1)$.

Proof. Immediate from the definition of ξ_1 in the proof of Lemma 3.

From Lemma 4 it follows that the mapping ξ_1 embeds $\mathrm{End}(L)$ isomorphically as a submonoid of $\mathrm{End}(L_1)$.

Lemma 5. *Every congruence* ϱ *on* L *is the restriction to* L *of some congruence* $\varrho^{(1)}$ *on* L_1, *where the mapping*

$$\zeta_1\colon \theta(L) \to \theta(L_1), \quad \varrho \to \varrho^{(1)}$$

is a lattice isomorphism of $\theta(L)$ *onto a closed sublattice of* $\theta(L_1)$.

Proof. If ϱ is any congruence on L, then we define the relation $\bar{\varrho}$ on $\mathscr{B}$ by

$$\bar{\varrho} = \{(eL\cup Y, fL\cup Y)|e, f\in L, \ e\varrho f, \ Y\in\mathscr{A}\}\cup\{(Y, Y)|Y\in\mathscr{A}\}.$$

Let $\varrho^{(1)}=\varphi_1\bar{\varrho}\varphi_1^{-1}$. It can be checked easily that $\bar{\varrho}$ and $\varrho^{(1)}$ are congruences on $\mathscr{B}$ and on L_1 respectively, and that ϱ is the restriction of $\varrho^{(1)}$ to L. Let us now consider the injective mapping $\zeta_1\colon \theta(L)\to\theta(L_1)$, $\varrho\to\varrho^{(1)}$. Let $\{\varrho_i|i\in I\}$ be any subset of $\theta(L)$. Clearly

$$\Big(\bigcap_{i\in I}\varrho_i\Big)\zeta_1 = \varphi_1\overline{\Big(\bigcap_{i\in I}\varrho_i\Big)}\varphi_1^{-1} = \varphi_1\Big(\bigcap_{i\in I}\bar{\varrho}_i\Big)\varphi_i^{-1} = \bigcap_{i\in I}\varphi_1\bar{\varrho}_i\varphi_1^{-1} = \bigcap_{i\in I}(\varrho_i\zeta_1).$$

Let A and B any elements of $\mathscr{B}$ such that

$$A\Big(\bigvee_{i\in I}\bar{\varrho}_i\Big)B.$$

Then there exist elements $A=A_0, \ldots, A_j, A_{j+1}, \ldots, A_k=B$ such that for every $j\in\{0, \ldots, k-1\}$, $A_j\bar{\varrho}_jA_{j+1}$ for some $\varrho_j\in\{\varrho_i|i\in I\}$. If A is of the form $A=Y$, $Y\in\mathscr{A}$, then $A=A_0=A_1=\ldots=A_k=B$. If A is of the form $eL\cup Y$, $e\in L$, $Y\in\mathscr{A}$, then A_j is of the form $e_jL\cup Y$ for all $j\in\{0, \ldots, k\}$, and for all $j\in\{0, \ldots, k-1\}$, $e_j\varrho_je_{j+1}$; thus B is then of the form $fL\cup Y$, where $e\ \overline{\big(\bigvee_{i\in I}\varrho_i\big)}\ f$ in L. We conclude that $\bigvee_{i\in I}\bar{\varrho}_i\subseteq \overline{\bigvee_{i\in I}\varrho_i}$. Similarly we can show that $\overline{\bigvee_{i\in I}\varrho_i}\subseteq\bigvee_{i\in I}\bar{\varrho}_i$. We conclude that $\overline{\bigvee_{i\in I}\varrho_i}=\bigvee_{i\in I}\bar{\varrho}_i$, and so

$$\Big(\bigvee_{i\in I}\varrho_i\Big)\zeta_1 = \varphi_1\overline{\Big(\bigvee_{i\in I}\varrho_i\Big)}\varphi_1^{-1} = \varphi_1\Big(\bigvee_{i\in I}\bar{\varrho}_i\Big)\varphi_1^{-1} = \bigvee_{i\in I}(\varphi_1\bar{\varrho}_i\varphi_1^{-1}) = \bigvee_{i\in I}(\varrho_i)\zeta_1.$$

Therefore ζ_1 is a lattice isomorphism of $\theta(L)$ onto a closed sublattice of $\theta(L_1)$.

We are now in the position to prove our main theorem.

Theorem. *Every lattice* L *can be isomorphically embedded as a dual ideal into a uniform lattice* L' *in such a way that*

(i) *every partial isomorphism* α *of* L *can be extended to a partial isomorphism* α' *of* L' *such that the mapping*

$$\psi: T_L \to T_{L'}, \quad \alpha \to \alpha'$$

is an isomorphism of T_L *into* $T_{L'}$,

(ii) *every endomorphism [automorphism]* γ *of* L *can be extended to an endomorphism [automorphism]* γ' *of* L' *such that the mapping*

$$\xi: \mathrm{End}(L) \to \mathrm{End}(L'), \quad \gamma \to \gamma'$$

is an isomorphism of $\mathrm{End}(L)$ *into* $\mathrm{End}(L')$ *which induces an isomorphism of* $\mathrm{Aut}(L)$ *into* $\mathrm{Aut}(L')$,

(iii) *every congruence* ϱ *on* L *is the restriction to* L *of some congruence* ϱ' *on* L' *where the mapping*

$$\zeta: \theta(L) \to \theta(L'), \quad \varrho \to \varrho'$$

is a lattice isomorphism of $\theta(L)$ *onto a closed sublattice of* $\theta(L')$.

Proof. Let us consider the sequence of lattices

$$L = L_0, L_1, \ldots, L_j, L_{j+1}, \ldots$$

where for every $j \in N$, L_{j+1} is a lattice which contains L_j as a dual ideal and where L_{j+1} is constructed from L_j in the same way as L_1 is constructed from $L=L_0$. Then $L' = \bigcup_{j=0}^{\infty} L_j$ is a lattice which contains each $L_j, j \in Z^+$ as a dual ideal; in particular L is a dual ideal of L'.

Let $\alpha^{(j)}$ be any partial isomorphism of L_j for some $j \in N$. Let us consider the sequence of partial isomorphisms

$$\alpha^{(j)}, \alpha^{(j+1)}, \ldots, \alpha^{(j+k)}, \alpha^{(j+k+1)}, \ldots$$

where for every $k \in N$, $\alpha^{(j+k+1)}$ is a partial isomorphism of L_{j+k+1} which extends the partial isomorphism $\alpha^{(j+k)}$ of L_{j+k} in the way prescribed by the proof of Lemma 2. Therefore $\bigcup_{k \in N} \alpha^{(j+k)} = \alpha'$ is a partial isomorphism of L' which extends $\alpha^{(j)}$.

Let us now consider any two elements $x, y \in L'$. There exists a $j \in Z^+$ such that $x, y \in L_{j-1}$. By Lemma 1 we know that there exists a partial isomorphism $\alpha^{(j)}$ of L_j which maps xL_j isomorphically onto yL_j. Let $\alpha' = \bigcup_{k \in N} \alpha^{(j+k)}$ be the partial isomorphism of L' which is obtained from $\alpha^{(j)}$ in the way described above. The partial isomorphism α' maps xL' isomorphically onto yL', and therefore $(x, y) \in \mathscr{U}_{L'}$. We conclude that L' is uniform.

If $\alpha=\alpha^{(0)}$ is any partial isomorphism of L, then $\alpha'=\bigcup_{j\in N}\alpha^{(j)}$ is a partial isomorphism of L' which extends α. Let us investigate the mapping $\psi\colon T_L\to T_{L'}$, $\alpha\to\alpha'$. If $\beta=\beta^{(0)}$ is any other partial isomorphism of L, then $\beta'=\bigcup_{j\in N}\beta^{(j)}=$ $=\beta\psi\in T_{L'}$, and it follows from Lemma 2 that for all $j\in N$, $\alpha^{(j)}\beta^{(j)}=(\alpha\beta)^{(j)}$. From this it follows that $\alpha'\beta'=(\alpha\beta)'$, and so ψ is an isomorphism of T_L into $T_{L'}$. We conclude that (i) is satisfied. Using Lemma 3 and Lemma 4 we can introduce an injective mapping $\xi\colon \mathrm{End}\,(L)\to\mathrm{End}\,(L')$, $\gamma\to\gamma'$ which satisfies (ii): the proof thereof proceeds along the same lines as for the foregoing case.

Let $\varrho=\varrho^{(0)}$ be any congruence on L, and let us consider the sequence of congruences

$$\varrho=\varrho^{(0)},\ \varrho^{(1)},\ \ldots,\ \varrho^{(j)},\ \varrho^{(j+1)},\ \ldots$$

where for every $j\in N$, $\varrho^{(j+1)}$ is a congruence on L_{j+1} which is constructed from $\varrho^{(j)}$ in the way prescribed by the proof of Lemma 5. It should be clear that for all $i,j\in N$, $i\leqq j$, we have $\varrho^{(j)}\cap L_i\times L_i=\varrho^{(i)}$. Furthermore $\varrho'=\bigcup_{j\in N}\varrho^{(j)}$ is a congruence on L', and the restriction of ϱ' to L is precisely ϱ. Let us investigate the injective mapping $\zeta\colon\theta(L)\to\theta(L')$, $\varrho\to\varrho'$. Let $\{\varrho_i|i\in I\}$ be any subset of $\theta(L)$. Clearly by Lemma 5 we have

$$\Big(\bigcap_{i\in I}\varrho_i\Big)\zeta=\Big(\bigcap_{i\in I}\varrho_i\Big)'=\bigcup_{j\in N}\Big(\bigcap_{i\in I}\varrho_i\Big)^{(j)}=\bigcup_{j\in N}\Big(\bigcap_{i\in I}\varrho_i^{(j)}\Big)=$$
$$=\bigcap_{i\in I}\Big(\bigcup_{j\in N}\varrho_i^{(j)}\Big)=\bigcap_{i\in I}\varrho_i'=\bigcap_{i\in I}(\varrho_i)\zeta.$$

Let us consider $\big(\bigvee_{i\in I}\varrho_i\big)\zeta=\big(\bigvee_{i\in I}\varrho_i\big)'$, and let us suppose that x and y are any elements of L' such that $x\,\big(\bigvee_{i\in I}\varrho_i\big)'\,y$. There exists a $j\in N$ such that $x,y\in L_j$. Since the restriction of $\big(\bigvee_{i\in I}\varrho_i\big)'$ to L_j is precisely $\big(\bigvee_{i\in I}\varrho_i\big)^{(j)}$, and since by Lemma 5 $\big(\bigvee_{i\in I}\varrho_i\big)^{(j)}=$ $=\bigvee_{i\in I}\varrho_i^{(j)}$, we must have $x\,\big(\bigvee_{i\in I}\varrho_i^{(j)}\big)\,y$. From $\big(\bigvee_{i\in I}\varrho_i^{(j)}\big)\subseteqq\big(\bigvee_{i\in I}\varrho_i'\big)$ it then follows that $x\big(\bigvee_{i\in I}\varrho_i'\big)\,y$. We conclude that $\big(\bigvee_{i\in I}\varrho_i\big)'\subseteqq\big(\bigvee_{i\in I}\varrho_i'\big)$. Let us conversely suppose that x and y are elements of L' such that $x\,\big(\bigvee_{i\in I}\varrho_i'\big)\,y$. Then there exist elements $x=x_0, x_1, \ldots$ $\ldots, x_k=y$ in L' such that for every $j\in\{0,\ldots,k-1\}$, $x_j\varrho_j'x_{j+1}$, $\varrho_j\in\{\varrho_i|i\in I\}$. There exists some $n\in N$ such that $\{x_0,\ldots,x_k\}\subseteqq L_n$, and then $x_j\varrho_j^{(n)}x_{j+1}$ for every $j\in\{0,\ldots,k-1\}$. Therefore $x\ \big(\bigvee_{i\in I}\varrho_i^{(n)}\big)\ y$, and by Lemma 5 we have $\big(\bigvee_{i\in I}\varrho_i^{(n)}\big)=$ $=\big(\bigvee_{i\in I}\varrho_i\big)^{(n)}$. Clearly $\big(\bigvee_{i\in I}\varrho_i\big)^{(n)}\subseteqq\big(\bigvee_{i\in I}\varrho_i\big)'$, and so $x\ \big(\bigvee_{i\in I}\varrho_i\big)'\ y$. We conclude that $\big(\bigvee_{i\in I}\varrho_i\big)\zeta=\big(\bigvee_{i\in I}\varrho_i\big)'=\bigvee_{i\in I}\varrho_i'=\bigvee_{i\in I}(\varrho_i)\zeta$. Thus the mapping ζ satisfies (iii). This concludes the proof of the theorem.

Remark. The concepts "partial isomorphism", "Munn semigroup", "uniform" were originally introduced for semilattices. The results of this paper remain valid if we deal with semilattices only; if we do so several simplifications in our construction may be conceived. Anyhow, our main theorem still holds if L and L' are semilattices; L is then embedded as a dual ideal in the uniform semilattice L' in such a way that (i), (ii) and (iii) are satisfied. That every semilattice can be embedded as a subsemilattice in a uniform semilattice also follows from Reilly's results in [4].

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The algebraic representation of semigroups and lattices; representing subsemigroups

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A monoid S and a lattice L are *jointly algebraic*, if there is a universal algebra $\mathfrak{A}=\langle A,\mathscr{P}\rangle$ such that $S\cong\operatorname{End}\mathfrak{A}$ and $L\cong\operatorname{Su}\mathfrak{A}$. The major result of this paper is that if either S or L are finite and if they are jointly algebraic, then every submonoid T of S is jointly algebraic with L. We prove a slightly stronger theorem.

§ 1. Introduction

We adopt the notation of [1] and [2]. If M is a set of partial functions on the set A then we will write sometimes $M^{\sim}$ for $\tilde{M}$ and we will use the following additional notation: Γ, Σ denote systems of equations with coefficients from M. For $D\subset A$, $\bar{D}=\mathscr{C}(D;A,M)=\bigcap_{D\subset\operatorname{Spt}\Sigma}$ ($\operatorname{Spt}\Sigma$ is the set of all points on which Σ has a solution). We write simply $\bar{D}$ if A and M are understood. For $B\subset A$, $\mathscr{S}B=\mathscr{S}(B;A,M)=\bigcup_{D\text{ finite},\,D\subset B}\bar{D}$. We write $\mathscr{S}B$ if A and M are understood. If $D\subset B$ and D is finite we will henceforth write $D\subset_f B$.

§ 2. Concrete Results

Lemma 1. *If $\mathfrak{A}$ is any algebra on A whose operations are all substitutive with M and Σ is a system of equations over M, then* $\operatorname{Spt}\Sigma$ *is a subalgebra of* $\mathfrak{A}$.

Proof. It is enough to prove that $\operatorname{Spt}\Sigma$ is a subalgebra of $\mathfrak{A}_M$, the algebra of all the operations substitutive over M. According to [1] Theorem 1 we have to show that $\operatorname{Spt}\Sigma=\mathscr{S}(\operatorname{Spt}\Sigma;A,M)$. Now if $D\subset\operatorname{Spt}\Sigma$, then $\bar{D}=\bigcap_{D\subset\operatorname{Spt}\Gamma}\operatorname{Spt}\Gamma\subset\operatorname{Spt}\Sigma$ and

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therefore $\mathcal{S}(\mathrm{Spt}\,\Sigma; A, M)=\bigcup_{D\subset_f \mathrm{Spt}\,\Sigma} \subset \bigcup \mathrm{Spt}\,\Sigma=\mathrm{Spt}\,\Sigma$. By [1] Lemma 5, $\mathcal{S}$ is a closure operator and hence $\mathrm{Spt}\,\Sigma\subset \mathcal{S}(\mathrm{Spt}\,\Sigma; A, M)$. □

Lemma 2. *If* $D\subset_f A$ *then* $\bar{D}$ *is the subalgebra of* $\mathfrak{A}_M$ *generated by* D *and* $\bar{D}=\mathrm{dom}\, g$ *for some partial identity function* $g\in\tilde{M}$.

Proof. Let B be the subalgebra of $\mathfrak{A}_M$ generated by D. Then by Theorem 1 of [1] $B=\bigcup_{C\subset_f B}\bar{C}$, thus $\bar{D}\subset B$. But also $\bar{D}=\bigcap_{D\subseteq \mathrm{Spt}\,\Sigma}\mathrm{Spt}\,\Sigma=\mathrm{Spt}\,\Gamma$ for some system Γ by Lemma 2 of [1], and hence $\bar{D}\in \mathrm{Su}\,\mathfrak{A}_M$ by Lemma 1 above. Thus $\bar{D}=B$. Clearly $\bar{D}=\mathrm{dom}\, g$ for $g=\mathrm{id}\restriction \mathrm{Spt}\,\Gamma$*, and since $\mathrm{Spt}\,\Gamma\in \mathrm{Su}\,\mathfrak{A}_M$ we have $g\in\tilde{M}$ because every identity on a subalgebra of $\mathfrak{A}_M$ is a partial endomorphism. □

Lemma 3. *If* D *is finite, then* $\mathcal{C}(D; A, M)=\mathcal{C}(D; A, \tilde{M})$.

Proof. Note $\mathfrak{A}_M=\mathfrak{A}_{\tilde{M}}$, hence the subalgebra generated by D is the same in both algebras and the result follows from Lemma 2 above. □

Corollary 1. $\mathcal{S}(B; A, M)=\bigcup_{D\subset_f B}\mathcal{C}(D; A, M)=\bigcup_{D\subset_f B}\mathcal{C}(D; A, \tilde{M})=\mathcal{S}(B; A, \tilde{M})$.

Definition 1. We will write the *ordered triple* $(A; S, L)$ for a representation of S as a transformation monoid on A and L as an algebraic intersection structure on A (i.e. L is a set of subsets of A, which forms by intersection an algebraic lattice). Then $\mathrm{St}_1(A; S, L)$ and $\mathrm{St}_2(A; S, L)$ are abbreviations for the following statements:

$$\mathrm{St}_1(A; S, L):\ S\Rightarrow \overline{S\cup L},$$

(where if M is a set of partial functions on A, $\bar{M}$ is the set of total functions in $\tilde{M}$) and

$$\mathrm{St}_2(A; S, L):\ B=\mathcal{S}(B; A, S\cup L)\Rightarrow B\in L.$$

If $(A; S, L)$ and $S=\mathrm{End}\,\mathfrak{A}$ and $L=\mathrm{Su}\,\mathfrak{A}$ for some algebra $\mathfrak{A}=\langle A, \mathcal{P}\rangle$ then we will say that $(A; S, L)$ is *algebraic*.

Remark. Then Theorem 3 of [1] reads (using also Theorem 4 of [2]): $(A; S, L)$ is algebraic if and only if $\mathrm{St}_1(A; S, L)$ and $\mathrm{St}_2(A; S, L)$.

Lemma 5. *If* $\mathrm{St}_2(A; S, L)$, *then* $(A; \overline{S\cup L}, L)$ *is algebraic.*

Proof. a) $\mathrm{St}_1(A, \overline{S\cup L}, L)$. Note that $\overline{S\cup L}\subset\overline{\overline{S\cup L}\cup L}$. Put on the other hand $\overline{\overline{S\cup L}\cup L}=A^A\cap[(A^A\cap(S\cup L)^{\sim})\cup L]^{\sim}\subset A^A\cap[(S\cup L)^{\sim}\cup L]^{\sim}=A^A\cap(S\cup L)^{\sim}=\overline{S\cup L}$. This proves $\mathrm{St}_1(A; \overline{S\cup L}, L)$ which says: $\overline{S\cup L}=\overline{\overline{S\cup L}\cup L}$.

* For a function f and $A\subseteq \mathrm{dom} f$, $f\restriction A$ denotes the restriction of f to A.

b) $\mathrm{St}_2(A, \overline{S \cup L}, L)$. We have $(S\cup L)^{\sim}=(\overline{S\cup L}\cup L)^{\sim}$ because obviously $(S\cup L)^{\sim}\subset(\overline{S\cup L}\cup L)^{\sim}$ and $(\overline{S\cup L}\cup L)^{\sim}=[(A^A\cap(S\cup L)^{\sim})\cup L]^{\sim}\subset((S\cup L)^{\sim}\cup L)^{\sim}=$ $=(S\cup L)^{\sim}$. Therefore, by Corollary 1, $\mathscr{S}(B; A, \overline{S\cup L}\cup L)=\mathscr{S}(B; A(\overline{S\cup L}\cup L)^{\sim})=$ $=\mathscr{S}(B; A, (S\cup L)^{\sim})=\mathscr{S}(B; A, S\cup L)$. So, if $B=\mathscr{S}(B; A, \overline{S\cup L}\cup L)$, then $B=$ $=\mathscr{S}(B; A, S\cup L)$ and hence $B\in L$ because $\mathrm{St}_2(A; S, L)$ holds. □

§ 3. Representations

Definition 2. If S is a monoid and L an algebraic lattice, then the partial universal algebra $\langle A; f\rangle_{f\in S\cup L}$ is a *representation* of S and L, if all of the operations in S form a transformation monoid of A, with $(fg)(a)=f(g(a))$, $\mathrm{id}(a)=a$ and if all of the operations in L are partial identities with range $p\cap$range $q=$range$(p\wedge q)$ and the 1 of the lattice is the identity transformation of A. Furthermore we require that a representation be faithful: for any two $f, g\in S, f\neq g$ there exists an $a\in A$ with $f(a)\neq g(a)$ and if for any two $p, q\in L, p\neq q$, range $p\neq$ range q. We write simply $\langle A, f\rangle$ for $\langle A, f\rangle_{f\in S\cup L}$ when $S\cup L$ is understood. Note $(A, \{f; f\in S\}, \{f(A); f\in L\})$ iff $\langle A, f\rangle_{f\in S\cup L}$ is a representation.

We will adopt the notions of [3] for homomorphism, subalgebra, embedding of partial algebras and will also say that $\mathfrak{B}$ is an extension of $\mathfrak{A}$ if $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$.

Definition 3. If $\langle A; f\rangle_{f\in S\cup L}$ is a representation of S and L then we will write $\overline{S\cup L}^A$ to emphasize the function closure cited in St_1 taken with respect to that representation of S and L on A.

Lemma 6. *Let $\psi\colon A\to B$ be a homomorphism from the representation $\langle A; f\rangle_{f\in S\cup L}$ into the representation $\langle B; f\rangle_{f\in S\cup L}$. If the system Σ of equations with coefficients in $S\cup L$ has a solution h at some $a\in A$, then Σ has also a solution ψh at $\psi(a)\in B$.*

Proof. If α is an assignment which satisfies Σ at a, then clearly $\psi\alpha$ is an assignment which satisfies Σ at $\psi(a)$. □

Definition 4. Let $\langle A; f\rangle_{f\in S\cup L}$ be a representation of S and L and let $(A_i\colon i\in I)$ be a family of subalgebras of A with $\bigcup_{i\in I} A_i=A$ and $(\varphi_i\colon i\in I)$ homomorphisms from A onto A_i which leave A_i elementwise fixed. $((A_i, \varphi_i)\colon i\in I)$ is called a *cover* of $\langle A; f\rangle_{f\in S\cup L}$.

Lemma 7. *If $((A_i, \varphi_i)\colon i\in I)$ is a cover of $\langle A; f\rangle_{f\in S\cup L}$ and $h\in\overline{S\cup L}^A$, then $h=\bigcup_{i\in I} h_i$ with each $h_i\in\overline{S\cup L}^{A_i}$.*

Proof. We prove that $h\restriction A_i\in\overline{S\cup L}^{A_i}$. For each $a\in A_i$ there exists a system Σ whose unique solution at a is h. If $h(a)\notin A_i$, then by applying φ_i and Lemma 6 we observe that Σ has a solution at $\varphi_i(a)=a$ which is equal to $\varphi_i(h(a))\in A_i$. But then Σ has two different solutions at a, a contradiction. So $h\upharpoonright A_i\in A_i^{A_i}$. Because $h\in\overline{S\cup L}^A$ there exists for every finite subset D of A_i a system Σ whose unique solution at D is h. We will have proven that $h\upharpoonright A_i\in\overline{S\cup L}^{A_i}$ if there exists an assignment α of Σ at D for $D\subset_f A_i$ with $\alpha(x)\in A_i$ for all the variables $x\in\Sigma$. If β is any assignment of Σ at D, then clearly $\varphi_i\beta=\alpha$ has the desired property. Thus $h=\bigcup_{i\in I} h_i$ for $h_i=h\upharpoonright A_i$. □

Lemma 8. *If* $((A_i,\varphi_i): i\in I)$ *is a cover of* $\langle A;f\rangle$ *and* $h\in\overline{S\cup L}^A$ *and* $x\in A_i\cap A_j$, *then* $h(x)\in A_i\cap A_j$.

Proof. According to Lemma 7, $h\upharpoonright A_i\in A_i^{A_i}$ and $h\upharpoonright A_j\in A_j^{A_j}$ which implies the assertion. □

§ 4. The Foliation of a Representation

Definition 5. If $\mathscr{R}=\langle A;f\rangle$ is a representation of S and L, and $a\in A$, then $[a]$ is the subalgebra of $\mathscr{R}$ generated by $\{a\}$.

Definition 6. If $\mathscr{R}=\langle A;f\rangle$ is a representation of S and L, then $\mathscr{F}(\mathscr{R})=\langle\mathscr{F}(A),f\rangle_{f\in S\cup L}$, the *foliation* of $\mathscr{R}$, is an extension of $\mathscr{R}$ which is constructed as follows: for each $x\in A$, $A_x^-=\{a_x;\, a\in A-[x]\}$ and $A_x=A_x^-\cup[x]$,

$$\mathscr{F}(A)=\bigcup_{x\in A} A_x;$$

with $y\in\mathscr{F}(A)$ and $f\in S$,

$$f(y)=\begin{cases} f(y) & \text{if } y\in A,\\ (f(a))_x & \text{if } y=a_x \text{ and } f(a)\in A-[x],\\ f(a) & \text{if } y=a_x \text{ and } f(a)\in[x];\end{cases}$$

with $y\in\mathscr{F}(A)$ and $p\in L$,

$$p(y)=\begin{cases} p(y) & \text{if } y\in A,\\ (p(a))_x & \text{if } y=a_x.\end{cases}$$

($p(y)$ is either y or is undefined).

Lemma 9. *$\mathscr{F}(\mathscr{R})$ is a representation of S and L.*

Proof. Let $f,g,h\in S$ with $(fg)=h$. We want to prove that for all $y\in\mathscr{F}(A)$, $f(g(y))=h(y)$. This is clearly true for $y\in A$, so let $y=a_x$ and assume first that $g(a)\notin[x]$ and $f(g(a))\notin[x]$, so $h(a)\notin[x]$ and then: $f(g(y))=f(g(a_x))=f((g(a))_x)=((fg)(a))_x=(h(a))_x=h(a_x)=h(y)$. If $g(a)\notin[x]$ but $f(g(a))\in[x]$, then $h(a)\in[x]$

and then: $f(g(y))=f((g(a))_x)=f(g(a))=(fg)(a)=h(a)=h(a_x)=h(y)$. If $g(a)\in[x]$, then $f(g(a))\in[x]$ and $h(a)\in[x]$ and then: $f(g(y))=f(g(a_x))=f(g(a))=(fg)(a)==h(a)=h(a_x)=h(y)$.

If id is the unit element in S and $y\in\mathcal{F}(A)$, then if $y\in A$ clearly $\mathrm{id}(y)=y$ and if $y=a_x$ then $a\notin[x]$ and $\mathrm{id}(a)=a\notin[x]$ and hence $\mathrm{id}(a_x)=(\mathrm{id}(a))_x=a_x$.

Let p, q be two elements in L, then $p(a_x)$ is defined and equal to a_x if and only if $p(a)$ is defined. But (on A), range $p\cap$range q=range $(p\wedge q)$ is equivalent to the condition: $(\forall a\in A)\,[p(a)$ and $q(a)$ are defined iff $(p\wedge q)(a)$ is defined]. Therefore $p(a_x)$ and $q(a_x)$ are defined iff $(p\wedge q)(a_x)$ is defined. Furthermore we have shown that the identity map on A extends to the identity map on $\mathcal{F}(A)$. Observe that $\mathcal{F}(\mathcal{R})$ is faithful iff $\mathcal{R}$ is faithful. □

Definition 7. If $\mathcal{F}(\mathcal{R})=\langle\mathcal{F}(A);f\rangle_{f\in S\cup L}$ is the foliation of $\mathcal{R}=\langle A;f\rangle_{f\in S\cup L}$, then the maps φ, $(\varphi_x;\, x\in A)$, $(\varepsilon_x;\, x\in A)$, $(\nu_x;\, x\in A)$ are defined as follows:

$$\nu_x\colon \mathcal{F}(A)\to A\cup A_x \text{ with } \nu_x(y)=\begin{cases} y & \text{if } y\in A\cup A_x, \\ a & \text{if } y=a_z,\ z\neq x; \end{cases}$$

$$\varepsilon_x\colon A\cup A_x\to A_x \text{ with } \varepsilon_x(y)=\begin{cases} y & \text{if } y\in A_x, \\ y_x & \text{if } y\in A-[x]; \end{cases}$$

$$\varphi\colon \mathcal{F}(A)\to A \text{ with } \varphi(y)=\begin{cases} y & \text{if } y\in A, \\ a & \text{if } y=a_x; \end{cases}$$

$$\varphi_x\colon \mathcal{F}(A)\to A_x \text{ with } \varphi_x=\varepsilon_x\nu_x.$$

Lemma 10. *Each of the maps above is a homomorphism onto the indicated subalgebra of* $\mathcal{F}(\mathcal{R})$.

Proof. a) ν_x. Because $A\cup A_x$ is a subalgebra, the restriction of ν_x to $A\cup A_x$ is a homomorphism. First let $a_z\in A_z$ and $f\in S$ with $f(a)\notin[z]$. Then $\nu_x(f(a_z))==\nu_x((f(a))_z)=f(a)=f(\nu_x(a_z))$. If $f(a)\in[z]$, then $\nu_x(f(a_z))=\nu_x(f(a))=f(a)=f(\nu_x(a_z))$. For $p\in L$, p is defined at a_z iff p is defined at a and $p(a_z)=a_z$ and $p(a)=a$, hence $p(\nu_x(a_z))=p(a)=a=\nu_x(a_z)=\nu_x(p(a_z))$.

b) φ. This proof is almost identical to the one for ν_x.

c) ε_x. Because A_x is a subalgebra of the representation $\langle A\cup A_x, f\rangle$ of S and L, the restriction of ε_x to A_x is a homomorphism. So if $y\in A-[x]$ and $f\in S$, with $f(y)\notin[x]$, then $\varepsilon_x f(y)=f(y)_x=f(y_x)=f(\varepsilon_x(y))$; further if $f(y)\in[x]$, then $\varepsilon_x f(y)=f(y)=f(y_x)=f(\varepsilon_x(y))$. If $p\in L$ and p is defined at $y\in A_x^-$, then p is defined at y_x and $p(y)=y, p(y_x)=y_x$ hence $p(\varepsilon_x(y))=p(y_x)=y_x=\varepsilon_x(y)=\varepsilon_x(p(y))$.

d) φ_x. φ_x is a homomorphism as a product of two homomorphisms. □

Lemma 11. *The sets $(A_x\colon x\in A)$ together with A and the maps $(\varphi_x\colon x\in A)$ and φ form a cover of* $\mathcal{F}(\mathcal{R})=\langle\mathcal{F}(A);f\rangle_{f\in S\cup L}$.

Proof. Clearly $(\bigcup_{x\in A} A_x)\cup A=\mathscr{F}(A)$ and furthermore A and each of the sets A_x are subalgebras of $\mathscr{F}(\mathscr{R})$. By Lemma 10 the maps φ, and $(\varphi_x\colon x\in A)$ are homomorphisms which leave A and $(A_x\colon x\in A)$ pointwise fixed as required. □

Lemma 12. $\bigcup_{x\in A} \varphi_x(\mathscr{C}(\varphi(D); A, S\cup L))=\mathscr{C}(D\colon \mathscr{F}(A), S\cup L)$, *for* $D\subseteq\mathscr{F}(A)$.

Proof. If $y\in A$ then $\varphi(y)=y$ and if $y=a_x$, then $\varphi_x\varphi(y)=\varphi_x\varphi(a_x)=$ $=\varepsilon_x v_x\varphi(a_x)=a_x=y$ and hence we see by Lemma 6 that a system Σ of equations has a solution at y iff Σ has a solution at $\varphi(y)$. Furthermore Σ has a solution at $y\in A$ iff Σ has a solution at $\varphi_x(y)$ for each $x\in A$, because again $\varphi\varphi_x(y)=y$. (For $y\in A\colon y\in[x]\Rightarrow\varphi\varphi_x(y)=y$, and $y\notin[x]\Rightarrow\varphi\varphi_x(y)=y$.) This means that $D\subset\operatorname{Spt}\Sigma$ iff $\varphi(D)\subset\operatorname{Spt}\Sigma$. In fact for $a\in A$, $a\in\operatorname{Spt}\Sigma$ iff $\forall x\in A, a\notin[x], a_x\in\operatorname{Spt}\Sigma$, thus $\bigcup_{x\in A}\varphi_x(A\cap\operatorname{Spt}\Sigma)=\operatorname{Spt}\Sigma$. Hence

$$\mathscr{C}(D,\mathscr{F}(A), S\cup L) = \bigcap_{D\subset\operatorname{Spt}\Sigma}\operatorname{Spt}\Sigma = \bigcap_{\varphi(D)\subset\operatorname{Spt}\Sigma}\operatorname{Spt}\Sigma =$$
$$= \bigcap_{\varphi(D)\subset\operatorname{Spt}\Sigma}\Big(\bigcup_{x\in A}\varphi_x(A\cap\operatorname{Spt}\Sigma)\Big) \supset \bigcup_{x\in A}\Big(\bigcap_{\varphi(D)\subset\operatorname{Spt}\Sigma}\varphi_x(A\cap\operatorname{Spt}\Sigma)\Big) \supset$$
$$\supset \bigcup_{x\in A}\varphi_x\Big(\bigcap_{\varphi(D)\subset\operatorname{Spt}\Sigma}(A\cap\operatorname{Spt}\Sigma)\Big) = \bigcup_{x\in A}\varphi_x\Big(\bigcap_{\varphi(D)\subset\operatorname{Spt}^*\Sigma}\operatorname{Spt}^*\Sigma\Big) =$$
$$= \bigcup_{x\in A}\varphi_x(\mathscr{C}(\varphi(D), A, S\cup L))$$

(cf. Lemma 6), where $\operatorname{Spt}^*\Sigma$ is the support in the original representation $\mathscr{R}=\langle A; f\rangle$.

On the other hand, because $\varphi_x=\varepsilon_x v_x$ is one-to-one on A, we get

$$\varphi_x(\mathscr{C}(\varphi(D); A, S\cup L)) = \varphi_x\Big(\bigcap_{\varphi(D)\subset\operatorname{Spt}^*\Sigma}\operatorname{Spt}^*\Sigma\Big) = \bigcap_{\varphi(D)\subset\operatorname{Spt}^*\Sigma}\varphi_x(\operatorname{Spt}^*\Sigma) =$$
$$= \bigcap_{\varphi(D)\subset\operatorname{Spt}\Sigma}\varphi_x(A\cap\operatorname{Spt}\Sigma) \subset \bigcap_{D\subset\operatorname{Spt}\Sigma}\operatorname{Spt}\Sigma = \mathscr{C}(D;\mathscr{F}(A), S\cup L). \qquad \square$$

Lemma 13. $\bigcup_{x\in A}\varphi_x(\mathscr{S}(\varphi(B); A, S\cup L))=\mathscr{S}(B;\mathscr{F}(A), S\cup L)$.

Proof. Observe that $\varphi(D)\subset_f\varphi(B)\Rightarrow\exists E\subset_f B$ such that $\varphi(E)=\varphi(D)$ hence

$$\bigcup_{x\in A}\varphi_x(\mathscr{S}(\varphi(B); A, S\cup L)) = \bigcup_{x\in A}\varphi_x\Big(\bigcup_{\varphi(D)\subset_f\varphi(B)}\mathscr{C}(\varphi(D); A, S\cup L)\Big) =$$
$$= \bigcup_{x\in A}\varphi_x\Big(\bigcup_{E\subset_f B}\mathscr{C}(\varphi(E); A, S\cup L)\Big) = \bigcup_{D\subset_f B}\Big(\bigcup_{x\in A}\varphi_x(\mathscr{C}(\varphi(D); A, S\cup L)\Big) =$$
$$= \bigcup_{D\subset_f B}\mathscr{C}(D;\mathscr{F}(A), S\cup L) = \mathscr{S}(B;\mathscr{F}(A), S\cup L). \qquad \square$$

Now by intersecting A with each of the expressions in Lemma 13 we have:

Corollary 2. $\mathscr{S}(\varphi(B); A, S\cup L)=\mathscr{S}(B;\mathscr{F}(A), S\cup L)\cap A$.

Definition 8. If $\mathscr{R}=\langle A;f\rangle$ is a representation of S and L, then we write $\mathrm{St}_2\,\mathscr{R}$ or $\mathrm{St}_2\,\langle A;f\rangle$ to mean St_2 holds for the corresponding triple (see Definition 2): $\mathrm{St}_2(A, \{f; f\in S\}, \{f(A); f\in L\})$.

Lemma 14. *If $\mathscr{R}=\langle A;f\rangle$ is a representation of S and L with $\mathrm{St}_2\,\mathscr{R}$, and $\mathscr{F}(\mathscr{R})=\langle\mathscr{F}(A),f\rangle$ is the foliation of $\mathscr{R}$, then $\mathrm{St}_2\,\mathscr{F}(\mathscr{R})$.*

Proof. Let $\mathscr{S}(B;\mathscr{F}(A), S\cup L)=B$; then $\mathscr{S}(\varphi(B); A, S\cup L)=\varphi(B)$ (otherwise $A\cap B\subset\varphi(B)\subsetneqq\mathscr{S}(\varphi(B); A, S\cup L)\subset\mathscr{S}(B;\mathscr{F}(A), S\cup L)\cap A=B\cap A$). Hence there is $p\in L$ with $\varphi(B)=\text{range}\,p$ in A. Then:

$$B=\mathscr{S}(B;\mathscr{F}(A), S\cup L)=\bigcup_{x\in A}\varphi_x(\mathscr{S}(\varphi(B); A, S\cup L))=\bigcup_{x\in A}\varphi_x\varphi(B)=$$

$$=\bigcup_{x\in A}\varepsilon_x\nu_x\varphi(B)=\bigcup_{x\in A}\varepsilon_x\varphi(B)=\bigcup_{x\in A}\varepsilon_x(\text{range } p \text{ in } A)=\text{range } p \text{ in } \mathscr{F}(A).$$

Thus $\mathrm{St}_2\,\mathscr{F}(\mathscr{R})$ holds. □

Lemma 15. *If $h\in\overline{S\cup L}^{\mathscr{F}(A)}$, then $m=h\restriction A\in\overline{S\cup L}^{A}$, and for all $a_x\in\mathscr{F}(A)$, $h(a_x)=(ma)_x$ if $m(a)\notin[x]$ and $h(a_x)=m(a)$ otherwise.*

Proof. By Lemma 7 and Lemma 11 $h=m\cup(\bigcup_{x\in A}h_x)$ with $m\in\overline{S\cup L}^{A}$, and $h\restriction a_x=h_x\in\overline{S\cup L}^{A_x}$. Now to each $a_x\in\mathscr{F}(A)$ there is a system Σ, such that h is the unique solution to Σ on $\{a, a_x\}$. Thus m is the unique solution to Σ at a and h_x is the unique solution to Σ at a_x. Note m is a solution to Σ at a and φ_x is a homomorphism, thus by Lemma 6, $\varphi_x m$ is a solution to Σ at $a_x=\varphi_x(a)$. But h is the unique solution to Σ at a_x, thus $h(a_x)=\varphi_x(ma)=\varepsilon_x\nu_x(ma)=\varepsilon_x(ma)$. Hence if $ma\notin[x]$, $h(a_x)=(ma)_x$ and if $ma\in[x]$, $h(a_x)=ma$. □

Corollary 3. *If $h\in\overline{S\cup L}^{\mathscr{F}(A)}$ and if $(h\restriction A)\in S$ on $\mathscr{R}$ then $h\in S$ on $\mathscr{F}(\mathscr{R})$.*

Definition 9. If $\mathscr{R}=\langle A;f\rangle$ is a representation of S and L and $h\in A^A$, then we write h is in the *one closure* of S in $\mathscr{R}$ (or shortly $h\in\mathrm{oc}\,(S)_{\mathscr{R}}$ or $h\in\mathrm{oc}(S)$) if for each $a\in A$ there exists $f\in S$ with $h(a)=f(a)$. Local closure of S is denoted by l.c.(S).

Lemma 16. *If $h\in\overline{S\cup L}^{\mathscr{F}(A)}$, then $m=(h\restriction A)$ is in the one closure of S in $\mathscr{R}$.*

Proof. Assume there is $a\in A$ such that for all $f\in S$ $f(a)\neq m(a)=h(a)$. Then there exists a system Σ of equations, whose unique solution at a is $m(a)\notin[a]$. The unique solution of Σ at $\varphi_a(a)=a$ is $\varphi_a(m(a))=(m(a))_a\neq m(a)$ which is a contradiction. □

Definition 10. The representation $\mathscr{R}=\langle A,f\rangle$ of S and L on A is *algebraic*, if the corresponding triple $(A; S, L)$ is *algebraic*.

Definition 11. oc $\mathscr{F}(\mathscr{R})=\langle\mathscr{F}(A);f\rangle_{f\in \mathrm{oc}(S)\cup L}$ where $\mathscr{R}=\langle A;f\rangle_{f\in S\cup L}$ a representation of S and L on A and the action of the operations in oc $(S)\cup L$ are as determined in $\overline{S\cup L}^{\mathscr{F}(A)}$.

Lemma 17. *If the representation $(A;\ S, L)$ has each compact $t\in L$ singleton generated, then $(\mathscr{F}(A);\ S, L)$ also has each compact $t\in L$ singleton generated.*

Proof. Observe that for all $a\in A$, we have for each $p\in L$ $\exists x[a_x\in p$ in $(\mathscr{F}(A);\ S, L)]$ iff $[a\in p$ on $(\mathscr{F}(A);\ S, L)]$ iff $\forall x[a_x\in p$ in $(\mathscr{F}(A);\ S, L)]$.

Lemma 18. *Let $(B;\ S, L)$ satisfy* St_2. *Suppose $a, b\in B$ are such that for every $p\in L$ $[a\in p\Rightarrow b\in p]$. Then each system of equations Σ over $S\cup L$ which has a solution at a also has a solution at b.*

Proof. Let Σ be a system of equations over $S\cup L$ which has a solution at a. Spt Σ denotes the set of all points in B on which Σ has a solution. Clearly $\mathrm{Spt}\,\Sigma=\bigcup_{D\subset_f \mathrm{Spt}\,\Sigma}\ \bigcap_{D\subseteq \mathrm{Spt}\,\Gamma} \mathrm{Spt}\,\Gamma=\mathscr{S}(\mathrm{Spt}\,\Sigma;\ B, S\cup L)$ hence by $\mathrm{St}_2\,(B;\ S, L)$, Spt $\Sigma\in L$.

Hence $b\in \mathrm{Spt}\,\Sigma$ as required. □

Lemma 19. *Given $(B;\ S, L)$ which satisfies* St_2 *and for which each compact $t\in L$ is singleton generated, if $h\in\overline{S\cup L}^{B}$ and $h\in \mathrm{oc}\,(S)$ on $(B;\ S, L)$ then $h\in \mathrm{l.c.}(S)$* on $(B;\ S, L)$.

Proof. Fix $\{b_1, \ldots, b_n\}\subset_f B$. Let $p\in L$ be generated by $\{b_1, \ldots, b_n\}$; thus p is compact, and there exists $b\in B$ which generates p as well. Let Σ be a system of equations with coefficients from $S\cup L$ such that h is the unique solution on $\{b, b_1, b_2, \ldots, b_n\}$. Since $h\in \mathrm{oc}\,(S)$ there is some $f\in S$ with $f(b)=h(b)$. Hence h is also the unique solution on $\{b\}$ to the system $\Gamma=\Sigma\cup\{fx_0=x_1\}$. By Lemma 18 Γ has also a solution on each b_i, $i=1, \ldots, n$. But $\Gamma\supseteq\Sigma$ so the solution to Γ on $\{b_1, \ldots, b_n\}$ is h. On the other hand $(fx_0=x_1)\in\Gamma$ hence the solution to Γ on $\{b_1, \ldots, b_n\}$ is f. Thus $f(b_i)=h(b_i)$ for $i=1, \ldots, n$, so $h\in \mathrm{l.c.}(S)$ as required. □

Lemma 20. *Let N be a monoid and L an algebraic lattice such that $(A;N, L)$ with* $\mathrm{St}_2\,(A;N, L)$, *then if S is a submonoid of N we have* $\mathrm{St}_2\,(A;\ S, L)$.

Proof. Clearly $\mathscr{C}(D;A, N\cup L)\subset\mathscr{C}(D;A, S\cup L)$ and hence for each $B\subset A$, $B\subset\mathscr{S}(B;A, N\cup L)\subset\mathscr{S}(B;A, S\cup L)$. So if $B=\mathscr{S}(B;A, S\cup L)$ we get $B=\mathscr{S}(B;A, N\cup L)$ and then $B\in L$ in $(A;N, L)$. □

Theorem 1. *If $(A;N, L)$ is algebraic and each compact $t\in L$ is singleton generated in that representation then for each submonoid $S\subseteq N$ we have $(\mathscr{F}(A);\ \mathrm{l.c.}(S), L)$ is algebraic, where $\mathrm{l.c.}(S)$ is the local closure of S in the representation $(\mathscr{F}(A);\ S, L)$.*

Proof. Let $(A; N, L)$ satisfy the hypothesis of the theorem and let S be a submonoid of N. By Lemma 20 $(A; S, L)$ satisfies St_2, and clearly each compact $t \in L$ is singleton generated in $(A; S, L)$ as well. By Lemmas 14 and 17 $(\mathscr{F}(A); S, L)$ also satisfies St_2 and each compact $t \in L$ is singleton generated in that representation. Furthermore by Lemma 5 $(\mathscr{F}(A); \overline{S \cup L}^{\mathscr{F}(A)}, L)$ is algebraic, and here again each compact $t \in L$ is singleton generated. We claim that $\overline{S \cup L}^{\mathscr{F}(A)} = \text{l.c.}(S)$, the local closure of S in $(\mathscr{F}(A); S, L)$; this will establish the result of the Theorem. Evidently $\overline{S \cup L}^{\mathscr{F}(A)} \supseteqq \text{l.c.}(S)$ so really only the other containment need be argued. Let $h \in \overline{S \cup L}^{\mathscr{F}(A)}$. Note $h \restriction A \in \text{oc}(S)$ in $(\mathscr{F}(A); S, L)$, since by Lemma 16 we have $m = h \restriction A \in \text{oc}(S)$ in $(A; S, L)$. In fact $h \in \text{oc}(S)$ in $(\mathscr{F}(A); S, L)$. To see that we need only check $h(a_x)$ for $a_x \in \mathscr{F}(A)$. If $h(a) \notin [x]$ we get $h(a_x) = (h(a))_x = (f(a))_x$ for some $f \in S$ and if $h(a) \in [x]$ we get $h(a_x) = ha = fa = f(a_x)$ for some $f \in S$ by use of Lemma 15 and the definition of action by S in $\mathscr{F}(A)$ (see Defn. 6). Now apply Lemma 19 with $(B; S, L) = (\mathscr{F}(A); S, L)$ to get $h \in \overline{S \cup L}^{\mathscr{F}(A)} \cap \text{oc}(S) \Rightarrow h \in \text{l.c.}(S)$ on $(\mathscr{F}(A); S, L)$ as required. □

Lemma 21. *The local closure of any finite monoid S is equal to S.*

Proof. Let the monoid S be represented on some set A and assume that $h \in$ local closure S and $h \notin S$. For each $f \in S$ let $a_f \in A$ be such that $h(a_f) \neq f(a_f)$ then $D = \{a_f; f \in S\}$ is finite and clearly $h \restriction D \neq f \restriction D$ for any $f \in S$, contrary to the selection of h in the local closure of S. Hence each h in local closure S also belongs to S. □

Theorem 2. *For each universal algebra $\mathfrak{A}$ there is a universal algebra $\mathfrak{B}$ satisfying $\mathrm{End}\,\mathfrak{A} \cong \mathrm{End}\,\mathfrak{B}$ and $\mathrm{Su}\,\mathfrak{A} = \mathrm{Su}\,\mathfrak{B}$; moreover every finitely generated subalgebra of $\mathfrak{B}$ is generated by a single element.*

Proof. Let $\mathfrak{A} = \langle A, F \rangle$, $S = \mathrm{End}\,\mathfrak{A}$ and $L = \mathrm{Su}\,\mathfrak{A}$. For any $C \subseteqq A$ we set $C^* = \bigcup_{n=1}^{\infty} C^n$. (Remark that we do not distinguish between C and C^1 and thus $C \subseteqq C^*$.) With any $\varphi \in S$ we associate a transformation $\varphi^* \colon A^* \to A^*$ defined by $\varphi^*((x_1, \ldots, x_k)) = (\varphi(x_1), \ldots, \varphi(x_k))$, $(x_1, \ldots, x_k) \in A^*$. Let $S^* = \{\varphi^* | \varphi \in S\}$ and $L^* = \{C^* | C \in L\}$. Then $S^* \cong S$ and $L^* \cong L$. We shall construct an algebra $\mathfrak{B} = \langle A^*, G \rangle$ such that $S^* = \mathrm{End}\,\mathfrak{B}$, $L^* = \mathrm{Su}\,\mathfrak{B}$ and every finitely generated subalgebra of $\mathfrak{B}$ is generated by a single element.

Let g_1, g_2 be unary operations and h a binary operation on A^* defined by the rules:

$$g_1((x_1, \ldots, x_k)) = x_1, \quad g_2((x_1, \ldots, x_k)) = (x_k, x_1, \ldots, x_{k-1})$$

and

$$h((x_1, \ldots, x_k), (y_1, \ldots, y_l)) = (x_1, \ldots, x_k, y_1, \ldots, y_l)$$

for every $(x_1, \ldots, x_k)$, $(y_1, \ldots, y_l) \in A^*$. Furthermore, with each operation $f \in F$ we associate an operation $f_{\mathfrak{B}}$ on A^* as follows. The arity of $f_{\mathfrak{B}}$ equals the one of f and $f_{\mathfrak{B}}$ is defined by

$$f_{\mathfrak{B}}((x_1^1, \ldots, x_{k_1}^1), \ldots, (x_1^n, \ldots, x_{k_n}^n)) = f(x_1^1, \ldots, x_1^n), \ (x_1^i, \ldots, x_{k_i}^i) \in A^*, \quad i = 1, \ldots, n.$$

Now set $G = \{f_{\mathfrak{B}} | f \in F\} \cup \{g_1, g_2, h\}$.

First consider End $\mathfrak{B}$. It is clear that $S^* \subseteq \operatorname{End} \mathfrak{B}$. Let $\Phi \in \operatorname{End} \mathfrak{B}$. If $x \in A$ then $\Phi(x) = \Phi(g_1(x)) = g_1(\Phi(x)) \in A$ showing that $\Phi \restriction A = \varphi \in A^A$. Furthermore, if $f \in F$ is n-ary and $x_1, \ldots, x_n \in A$, then $\varphi(f(x_1, \ldots, x_n)) = \Phi(f_{\mathfrak{B}}(x_1, \ldots, x_n)) = f_{\mathfrak{B}} \cdot (\Phi(x_1), \ldots, \Phi(x_n)) = f(\varphi(x_1), \ldots, \varphi(x_n))$, i.e. $\Phi \restriction A = \varphi \in \operatorname{End} \mathfrak{A} = S$. Now we show by induction on k that (1) $\Phi((x_1, \ldots, x_k)) = (\varphi(x_1), \ldots, \varphi(x_k))$, $(x_1, \ldots, x_k) \in A^*$. If $k = 1$ then (1) holds. Supppose (1) holds for $k-1$. Then $\Phi((x_1, \ldots, x_k)) = \Phi(h((x_1, \ldots, x_{k-1}), x_k)) = h(\Phi((x_1, \ldots, x_{k-1}), \Phi(x_k)) = h((\varphi(x_1), \ldots, \varphi(x_{k-1})), \varphi(x_k)) = (\varphi(x_1), \ldots, \varphi(x_k))$. Hence $\Phi = \varphi^* \in S^*$.

Now consider Su $\mathfrak{B}$. It is clear that $L^* \subseteq \operatorname{Su} \mathfrak{B}$. Let $B \in \operatorname{Su} \mathfrak{B}$. Taking into account that g_1, g_2 and h are operations of $\mathfrak{B}$, one can show that $B = (B \cap A)^*$. Furthermore, $B \cap A \in \operatorname{Su} \mathfrak{A} = L$. $B = (B \cap A)^* \in L^*$. Finally, if a subalgebra B of $\mathfrak{B}$ is generated by the elements $(x_1^1, \ldots, x_{k_1}^1), \ldots, (x_1^s, \ldots, x_{k_s}^s) \in A^*$ then B is also generated by $(x_1^1, \ldots, x_{k_1}^1, \ldots, x_1^s, \ldots, x_{k_s}^s) \in A^*$ which completes the proof. □

Corollary 4. *If the monoid N and the algebraic lattice L are jointly algebraic and S is a finite submonoid of N, then S and L are jointly algebraic.*

Proof. Let $(A; N, L)$ be algebraic, with each compact $t \in L$ singleton generated in that representation. By Theorem 1 $(\mathscr{F}(A); \text{l.c.}(S), L)$ is algebraic. By Lemma 21 l.c.$(S) = S$ since S is finite, hence $(\mathscr{F}(A); S, L)$ is algebraic and S and L are (abstractly) jointly algebraic. □

Corollary 5. *If $S \subset T$ are two monoids and if L is an algebraic lattice for which the highest element 1 is compact and if T and L are jointly algebraic, then S and L are jointly algebraic.*

Proof. Let $\mathfrak{A} = \langle A; \mathscr{P} \rangle$ be such that $L = \operatorname{Su} \mathfrak{A}$ and $T = \operatorname{End} \mathfrak{A}$. We may assume each compact $t \in L$ is singleton generated in $\mathfrak{A}$. For the triple $(A; T, L)$ given by $\mathfrak{A}$ we have $(\mathscr{F}(A); \text{l.c.}(S)., L)$ algebraic. In fact by Lemma 17 each compact $t \in L$ is singleton generated in this representation. In particular $1 \in L$ which is compact by hypothesis is singleton generated. It follows that l.c.$(S) = S$ in that representation, hence $(\mathscr{F}(A); S, L)$ is algebraic and S, L are (abstractly) jointly algebraic. □

Corollary 6. *If the monoid T and the algebraic lattice L are jointly algebraic but not both infinite then every submonoid of T is jointly algebraic with L.*

Proof. Follows now immediately from Corollaries 4 and 5. □

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Factorisations régulières et sous-espaces invariants

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1. Soient $\mathfrak{E}$ et $\mathfrak{E}_*$ deux espaces de Hilbert complexes séparables, soit $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ une fonction analytique contractive pure [1]) et soit T l'opérateur qui lui est associé par

$$T^*(u\oplus v) = e^{-it}(u(e^{it})-u(0))\oplus e^{-it}v(t)$$

sur l'espace $\mathfrak{H}=\mathfrak{K}_+\ominus\mathfrak{G}$ où $\mathfrak{K}_+=H^2(\mathfrak{E}_*)\oplus(\Delta L^2(\mathfrak{E}))^-$, $\mathfrak{G}=\{\Theta u\oplus\Delta u\colon u\in H^2(\mathfrak{E})\}$ et $\Delta(t)=[I-\Theta^*(e^{it})\cdot\Theta(e^{it})]^{1/2}$.

Envisageons une factorisation $\Theta=\Theta_2\Theta_1$ de $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$, où les facteurs $\{\mathfrak{E}, \mathfrak{F}, \Theta_1(\lambda)\}$ et $\{\mathfrak{F}, \mathfrak{E}_*, \Theta_2(\lambda)\}$ sont des fonctions analytiques contractives et soit Z le prolongement à $(\Delta L^2(\mathfrak{E}))^-$ de l'opérateur isométrique $Z_0\colon \Delta L^2(\mathfrak{E})\to$ $\to(\Delta_2 L^2(\mathfrak{F}))^-\oplus(\Delta_1 L^2(\mathfrak{E}))^-$ défini par $Z_0\colon \Delta v\to\Delta_2\Theta_1 v\oplus\Delta_1 v$, $v\in L^2(\mathfrak{E})$.

Rappelons que la factorisation $\Theta=\Theta_2\Theta_1$ est dite régulière si Z est un opérateur unitaire, *cf.* [H] ch. VII. On y trouve aussi les suivants résultats:

(a) A chaque sous-espace $\mathfrak{H}_1\subset\mathfrak{H}$ invariant pour l'opérateur T il correspond une factorisation régulière $\Theta=\Theta_2\Theta_1$ telle que le sous-espace $\mathfrak{H}_1$ et son complément orthogonal $\mathfrak{H}_2=\mathfrak{H}\ominus\mathfrak{H}_1$ ont les représentations suivantes:

$$\text{(1)}\quad \mathfrak{H}_1=\{\Theta_2 u\oplus Z^{-1}(\Delta_2 u\oplus v)\colon u\in H^2(\mathfrak{F}),\ v\in(\Delta_1 L^2(\mathfrak{E}))^-,\ \Theta_1^* u+\Delta_1 v\perp H^2(\mathfrak{E})\},$$

$$\text{(1')}\quad \mathfrak{H}_2=\{u\oplus Z^{-1}(v\oplus 0)\colon u\in H^2(\mathfrak{E}_*),\ v\in(\Delta_2 L^2(\mathfrak{F}))^-,\ \Theta_2^* u+\Delta_2 v\perp H^2(\mathfrak{F})\}.$$

(b) Pour toute factorisation régulière $\Theta=\Theta_2\Theta_1$ le sous-espace $\mathfrak{H}_1$ donné par la formule (1) est un sous-espace invariant pour T.

Nous rappelons aussi que si $S\in\mathscr{B}(\mathfrak{H})$ est un élément du commutant $\{T\}'=$ $=\{S\in\mathscr{B}(\mathfrak{H})\colon ST=TS\}$ alors (*cf.* [1]) il existe des fonctions analytiques bornées $\{\mathfrak{E}_*, \mathfrak{E}_*, A(\lambda)\}$, $\{\mathfrak{E}, \mathfrak{E}, A_0(\lambda)\}$, des fonctions mesurables bornées $B(\cdot)\colon \mathfrak{E}_*\to(\Delta\mathfrak{E})^-$, $C(\cdot)\colon(\Delta\mathfrak{E})^-\to(\Delta\mathfrak{E})^-$ liées par les équations

$$\text{(2)}\qquad A\cdot\Theta=\Theta\cdot A_0\quad\text{et}\quad B\Theta+C\Delta=\Delta\cdot A_0$$

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[1]) Pour toutes les notions qui ne sont pas explicitement définies ainsi que pour la notation utilisée *cf.* [H].

et telles qu'on ait $S=P_+Y|\mathfrak{H}$ où P_+ est la projection orthogonale de $\mathfrak{K}_+$ sur $\mathfrak{H}$ et Y est l'opérateur dans $\mathfrak{K}_+$ ayant par rapport à la décomposition $\mathfrak{K}_+=$ $=\mathfrak{H}\oplus\mathfrak{G}$ la matrice

$$Y=\begin{bmatrix} A & O \\ B & C\end{bmatrix}.$$

Le but de cette Note est de déterminer des conditions sur un élément S de $\{T\}'$ pour que le sous-espace $\mathfrak{H}_1$ soit invariant pour S. De même nous obtenons des conditions nécessaires pour que le sous-espace $\mathfrak{H}_1$ soit de la forme $\mathfrak{H}_1=(S\mathfrak{H})^-$ pour un $S\in\{T\}'$.

2. Dans ce paragraphe nous déterminons la structure de l'opérateur $S\in\{T\}'$ tel que le sous-espace $\mathfrak{H}_1$ soit invariant pour S. La factorisation $\Theta=\Theta_2\Theta_1$ étant fixée nous considérerons le modèle fonctionnel, unitairement équivalent à celui dejà introduit, donné par

$$\mathbf{T}^*(u\oplus v_2\oplus v_1)=e^{-it}\big(u(e^{it})-u(0)\big)\oplus e^{-it}v_2(t)\oplus e^{-it}v_1(t)$$

sur l'espace $\mathbf{H}=\mathbf{K}_+\ominus\mathbf{G}$ où $\mathbf{K}_+=H^2(\mathfrak{E}_*)\oplus(\Delta_2L^2(\mathfrak{F}))^-\oplus(\Delta_1L^2(\mathfrak{E}))^-$ et $\mathbf{G}=$ $=\{\Theta u\oplus\Delta_2\Theta_1u\oplus\Delta_1u\colon u\in H^2(\mathfrak{E})\}$; le sous-espace $\mathbf{H}_1$ correspondant à $\mathfrak{H}_1$ est alors donné par

$$\mathbf{H}_1=\big\{\Theta_2u\oplus\Delta_2u\oplus v\colon\ u\in H^2(\mathfrak{F}),\ v\in(\Delta_1L^2(\mathfrak{E}))^-,\ \Theta_1^*u+\Delta_1v\perp H^2(\mathfrak{E})\big\}.$$

Dans ce cas, si l'on note

$$ZB=\begin{bmatrix} B_2 \\ B_1\end{bmatrix}\quad\text{et}\quad ZCZ^{-1}=\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22}\end{bmatrix}$$

l'opérateur $\mathbf{S}\in\{\mathbf{T}\}'$ correspondant à S sera donné par $\mathbf{S}=\mathbf{P}_+\mathbf{Y}|\mathbf{H}$ où $\mathbf{P}_+$ est la projection orthogonale de $\mathbf{K}_+$ sur $\mathbf{H}$ et $\mathbf{Y}$ a la forme

$$\mathbf{Y}=\begin{bmatrix} A & O & O \\ B_2 & C_{11} & C_{12} \\ B_1 & C_{21} & C_{22}\end{bmatrix}\tag{3}$$

où

$$\begin{cases} A\Theta=\Theta A_0, \\ B_2\Theta+C_{11}\Delta_2\Theta_1+C_{12}\Delta_1=\Delta_2\Theta_1A_0, \\ B_1\Theta+C_{21}\Delta_2\Theta_1+C_{22}\Delta_1=\Delta_1A_0.\end{cases}\tag{2'}$$

Soit donc $\mathbf{S}=\mathbf{P}_+\mathbf{Y}|\mathbf{H}$ où $\mathbf{Y}$ est de forme (3). Rappelons le suivant

Lemme ([2] lemme 1). *Si le sous-espace* $\mathbf{H}_1$ *est invariant pour* $\mathbf{S}\in\{\mathbf{T}\}'$, *on a* $C_{12}=O$.

Nous allons prouver le suivant

Lemme 1. *Si le sous-espace* $\mathbf{H}_1$ *est invariant pour* $\mathbf{S}\in\{\mathbf{T}\}'$, *il existe une fonction analytique bornée* $\{\mathfrak{F}, \mathfrak{F}, F(\lambda)\}$ *telle que*

$$A\Theta_2 = \Theta_2 F \quad et \quad F\Theta_1 = \Theta_1 A_0. \tag{4}$$

Démonstration. Notons que $\mathbf{H}_1$ est invariant pour $\mathbf{S}\in\{\mathbf{T}\}'$ si

$$\mathbf{L} = \{\Theta_2 u \oplus \Delta_2 u \oplus v \colon u\in H^2(\mathfrak{F}),\ v\in(\Delta_1 L^2(\mathfrak{E}))^-\}$$

est invariant pour $\mathbf{Y}$ et dans ce cas seulement. Pour $u\in H^2(\mathfrak{F})$ on a $\Theta_2 u\oplus\Delta_2 u\oplus 0\in\mathbf{L}$ d'où $\mathbf{Y}(\Theta_2 u\oplus\Delta_2 u\oplus 0)\in\mathbf{L}$; il existe donc $w\in H^2(\mathfrak{F})$ tel que $A\Theta_2 u=\Theta_2 w$ et $B_2\Theta_2 u+ +C_{11}\Delta_2 u=\Delta_2 w$. On peut vérifier facilement que l'application $u\mapsto w$ est fermée donc continue et permute à la multiplication par e^{it}. Donc il existe une fonction analytique bornée $F(\lambda)$ telle que

$$A\Theta_2 = \Theta_2 F \quad \text{et} \quad B_2\Theta_2 + C_{11}\Delta_2 = \Delta_2 F. \tag{4'}$$

Pour vérifier la deuxième relation (4) notons que la relation $A\Theta_2=\Theta_2 F$ obtenue auparavant implique $\Theta_2(F\Theta_1-\Theta_1 A_0)=O$. En tenant compte de la deuxième relation (2'), de (4') et $C_{12}=O$ on a $\Delta_2 F\Theta_1=B_2\Theta+C_{11}\Delta_2\Theta_1=\Delta_2\Theta_1 A_0$ d'où $\Delta_2(F\Theta_1-\Theta_1 A_0)=O$. Les relations $\Theta_2(F\Theta_1-\Theta_1 A_0)=O$ et $\Delta_2(F\Theta_1-\Theta_1 A_0)=O$ impliquent $F\Theta_1=\Theta_1 A_0$.

Ainsi nous avons démontré la nécessité dans la suivante

Proposition 1. *Soit* $\mathbf{H}_1$ *le sous-espace invariant correspondant à la factorisation* $\Theta=\Theta_2\Theta_1$; *supposons de plus que* $\Theta_1^*(e^{it})$ *est injectif pour presque tout* $t\in[0, 2\pi]$. *Pour que le sous-espace* $\mathbf{H}_1$ *soit invariant à* $\mathbf{S}=\mathbf{P}_+\mathbf{Y}|\mathbf{H}$ *où* $\mathbf{Y}$ *est donné par* (3), *il faut et il suffit que les suivantes conditions soient vérifiées*:

(i) $C_{12}=O$, (ii) *il existe une fonction analytique bornée* $\{\mathfrak{F}, \mathfrak{F}, F(\lambda)\}$, *telle que* $A\Theta_2=\Theta_2 F$ *et* $F\Theta_1=\Theta_1 A_0$.

Nous allons démontrer que les conditions (i) et (ii) sont aussi suffisantes pour que le sous-espace $\mathbf{H}_1$ soit invariant pour $\mathbf{S}$, sous l'hypothèse que $\Theta_1^*(e^{it})$ est injectif pour presque tout $t\in[0, 2\pi]$. Soit pour cela $\mathbf{S}=\mathbf{P}_+\mathbf{Y}|\mathbf{H}\in\{\mathbf{T}\}'$; d'après (i) nous obtenons pour $\mathbf{Y}$ une matrice de la forme

$$\mathbf{Y} = \begin{bmatrix} A & O & O \\ B_2 & C_{11} & O \\ B_1 & C_{21} & C_{22} \end{bmatrix}.$$

Les relations (2') deviennent

$$\begin{cases} A\Theta = \Theta A_0, \\ B_2\Theta + C_{11}\Delta_2\Theta_1 = \Delta_2\Theta_1 A_0, \\ B_1\Theta + C_{21}\Delta_2\Theta_1 + C_{22}\Delta_1 = \Delta_1 A_0. \end{cases} \tag{2''}$$

Mais d'après (ii) il existe une fonction analytique bornée $\{\mathfrak{F}, \mathfrak{F}, F(\lambda)\}$, telle que $A\Theta_2=\Theta_2 F$ et $F\Theta_1=\Theta_1 A_0$. Eu égard aussi à la deuxième relation (2″) on obtient $(B_2\Theta_2+C_{11}\Delta_2)\Theta_1=\Delta_2 F\Theta_1$; comme $\Theta_1^*(e^{it})$ est injectif il en résulte $B_2\Theta_2+C_{11}\Delta_2=$ $=\Delta_2 F$. Les relations $A\Theta_2=\Theta_2 F$ et $B_2\Theta_2+C_{11}\Delta_2=\Delta_2 F$ démontrées auparavant montrent que le sous-espace $\mathbf{L}$ est invariant pour $\mathbf{Y}$ donc $\mathbf{H}_1$ est invariant pour $\mathbf{S}$.

3. Dans la suite nous envisageons le cas où le sous-espace invariant $\mathbf{H}_1$ est de la forme $\mathbf{H}_1=(\mathbf{S}(\mathbf{H}))^-$ où $\mathbf{S}=\mathbf{P}_+\mathbf{Y}|\mathbf{H}\in\{\mathbf{T}\}'$; notons que dans ce cas $C_{12}=O$ car le sous-espace $\mathbf{H}_1$ est invariant pour $\mathbf{S}$. On a le suivant

Lemme 2. *Si* $\mathbf{S}(\mathbf{H})\subset\mathbf{H}_1$ *alors* $C_{11}=O$.

En effet soit $v_2\in(\Delta_2 L^2(\mathfrak{F}))^-$. Alors

$$\mathbf{S}\mathbf{P}_+(0\oplus v_2\oplus 0) = \mathbf{P}_+\mathbf{Y}\mathbf{P}_+(0\oplus v_2\oplus 0) = \mathbf{P}_+\mathbf{Y}(0\oplus v_2\oplus 0)$$

d'où $\mathbf{Y}(0\oplus v_2\oplus 0)\in\mathbf{L}$. Donc il existe $w\in H^2(\mathfrak{F})$ tel que $0\oplus C_{11}v_2=\Theta_2 w\oplus\Delta_2 w$ donc $\Theta_2 w=0$ et $\Delta_2 w=C_{11}v_2$. La relation $\Theta_2 w=0$ implique $\Delta_2 w=w$ donc $C_{11}\colon(\Delta_2 L^2(\mathfrak{F}))^-\to H^2(\mathfrak{F})$; comme de plus C_{11} permute à la multiplication par e^{it}, on conclut que $C_{11}=O$.

Lemme 3. *Si* $\mathbf{S}(\mathbf{H})\subset\mathbf{H}_1$, *alors il existe une fonction analityque bornée* $\{\mathfrak{E}_*, \mathfrak{F}, \Phi(\lambda)\}$ *telle que* $A=\Theta_2\Phi$ *et* $B_2=\Delta_2\Phi$.

Démonstration. Soit $u\in H^2(\mathfrak{E}_*)$. On a $\mathbf{P}_+\mathbf{Y}(u\oplus 0\oplus 0)\in\mathbf{H}$ donc $\mathbf{Y}(u\oplus 0\oplus 0)=$ $=Au\oplus B_2 u\oplus B_1 u\in\mathbf{L}$; ainsi, il existe $w\in H^2(\mathfrak{F})$ tel que $Au=\Theta_2 w$ et $B_2 u=\Delta_2 w$. L'application $\Phi\colon u\mapsto w$ est linéaire, continue et permute à la multiplication par e^{it} donc il existe une fonction analytique bornée $\{\mathfrak{E}_*, \mathfrak{F}, \Phi(\lambda)\}$ telle que

$$A = \Theta_2\Phi \quad \text{et} \quad B_2 = \Delta_2\Phi.$$

Remarque 1. La fonction $\{\mathfrak{E}_*, \mathfrak{F}, \Phi(\lambda)\}$ vérifie la relation

$$\Phi\Theta = \Theta_1 A_0. \tag{5}$$

En effet vu que $C_{11}=O$ et $C_{12}=O$ les conditions $A\Theta=\Theta A_0$ et $B_2\Theta+C_{11}\Delta_2\Theta_1+$ $+C_{12}\Delta_1=\Delta_2\Theta_1 A_0$, impliquent $\Theta_2\Phi\Theta=\Theta_2\Theta_1 A_0$ et $\Delta_2\Phi\Theta=\Delta_2\Theta_1 A_0$, d'où (5).

Conséquence 1. *Si pour l'opérateur* $\mathbf{S}=\mathbf{P}_+\mathbf{Y}|\mathbf{H}\in\{\mathbf{T}\}'$ *on a* $\mathbf{S}(\mathbf{H})\subset\mathbf{H}_1$, *alors la matrice de* $\mathbf{Y}$ *a la forme*

$$\mathbf{Y} = \begin{bmatrix} \Theta_2\Phi & O & O \\ \Delta_2\Phi & O & O \\ B_1 & C_{21} & C_{22} \end{bmatrix}$$

où $\Phi\Theta=\Theta_1 A_0$ *et* $B_1\Theta+C_{21}\Delta_2\Theta_1+C_{22}\Delta_1=\Delta_1 A_0$.

Lemme 4. *Si l'opérateur* $\mathbf{S}=\mathbf{P}_+\mathbf{Y}|\mathbf{H}\in\{\mathbf{T}\}'$ *vérifie la condition* $\mathbf{S}(\mathbf{H})=\mathbf{H}_1$, *alors*

$$H^2(\mathfrak{F}) = \Phi H^2(\mathfrak{E}_*)+\Theta_1 H^2(\mathfrak{E}). \tag{6}$$

Démonstration. La relation $\mathbf{P}_+\mathbf{Y}(\mathbf{H})=\mathbf{H}_1$ implique $\mathbf{Y}(\mathbf{K}_+)+\mathbf{G}=\mathbf{L}$; donc pour tout $u\in H^2(\mathfrak{F})$ il existe $u'\in H^2(\mathfrak{E}_*)$, $w'\in H^2(\mathfrak{E})$, $v_2\in(\Delta_2 L^2(\mathfrak{F}))^-$ et $v_1\in(\Delta_1 L^2(\mathfrak{E}))^-$ telles que

$$\mathbf{Y}(u'\oplus v_2\oplus v_1)+\Theta w'\oplus\Delta_2\Theta_1 w'\oplus\Delta_1 w' = \Theta_2 u\oplus\Delta_2 u\oplus 0.$$

Pour les premières deux composantes on a

$$\Theta_2(\Phi u'+\Theta_1 w') = \Theta_2 u \quad \text{et} \quad \Delta_2(\Phi u'+\Theta_1 w') = \Delta_2 u$$

d'où $u=\Phi u'+\Theta_1 w'$ donc (6).

Remarque 2. La décomposition $u=\Phi u'+\Theta_1 w'$ donnée par le lemme précédent n'est pas unique. En effet soit$^0 n$ $\in H^2(\mathfrak{E})$; vu que $\Phi\Theta=\Theta_1 A_0$ on a

$$u = \Phi(u'+\Theta u_0)+\Theta_1(w'-A_0 u_0).$$

Lemme 4'. *Si l'opérateur* $\mathbf{S}=\mathbf{P}_+\mathbf{Y}|\mathbf{H}\in\{\mathbf{T}\}'$ *vérifie la condition* $(\mathbf{S}(\mathbf{H}))^-=\mathbf{H}_1$, *alors*

$$H^2(\mathfrak{F}) = (\Phi H^2(\mathfrak{E}_*)+\Theta_1 H^2(\mathfrak{E}))^-. \tag{6'}$$

Démonstration. En effet, vu que $\mathbf{H}_1=(\mathbf{P}_+\mathbf{Y}(\mathbf{K}_+))^-$ il en résulte que le sous-espace $\mathbf{P}_+\mathbf{Y}(\mathbf{K}_+)+\mathbf{G}$ est dense dans $\mathbf{H}_1\oplus\mathbf{G}$, donc $\mathbf{Y}(\mathbf{K}_+)+\mathbf{G}$ est dense dans $\mathbf{H}_1\oplus\mathbf{G}$. Soit $u\in H^2(\mathfrak{F})$; alors $\Theta_2 u\oplus\Delta_2 u\oplus 0\in\mathbf{H}_1\oplus\mathbf{G}$. Donc pour tout $\varepsilon>0$ il existe $u'\oplus v_2\oplus v_1\in\mathbf{K}_+$ et $\Theta w\oplus\Delta_2\Theta_1 w\oplus\Delta_1 w\in\mathbf{G}$ tels que

$$\|\mathbf{Y}(u'\oplus v_2\oplus v_1)+(\Theta w\oplus\Delta_2\Theta_1 w\oplus\Delta_1 w)-(\Theta_2 u\oplus\Delta_2 u\oplus 0)\| \leqq \varepsilon.$$

On a donc

$$\|\Theta_2(\Phi u'+\Theta_1 w-u)\|^2+\|\Delta_2(\Phi u'+\Theta_1 w-u)\|^2 \leqq \varepsilon^2$$

et par conséquent $\|\Phi u'+\Theta_1 w-u\|\leqq\varepsilon$, d'où (6').

En combinant les lemmes 3 et 4' et la remarque 1 nous obtenons le

Theorème. *Soit* $T\in\mathscr{B}(\mathfrak{H})$ *une contraction complètement non-unitaire de l'espace de Hilbert* $\mathfrak{H}$, *dont la fonction caractéristique* $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ *admet la factorisation régulière* $\Theta=\Theta_2\Theta_1$, *où les facteurs* $\{\mathfrak{E}, \mathfrak{F}, \Theta_1(\lambda)\}$ *et* $\{\mathfrak{F}, \mathfrak{E}_*, \Theta_2(\lambda)\}$ *sont des fonctions analytiques bornées. Envisageons le sous-espace* $\mathfrak{H}_1$ *de* $\mathfrak{H}$ *invariant pour* T *correspondant à cette factorisation. Si pour un opérateur* $S\in\{T\}'$ *on a* $(S(\mathfrak{H}))^-=\mathfrak{H}_1$, *il existe des fonctions analytiques bornées* $\{\mathfrak{E}_*, \mathfrak{F}, \Phi(\lambda)\}$ *et* $\{\mathfrak{G}, \mathfrak{E}, A_0(\lambda)\}$, *telles que*

(i) $\Phi\Theta=\Theta_1 A_0$ *et* (ii) $H^2(\mathfrak{F})=(\Phi H^2(\mathfrak{E}_*)+\Theta_1 H^2(\mathfrak{E}))^-$.

4. Dans ce qui suit nous allons faire usage du théorème précédent pour démontrer la suivante

Proposition 2. *Soit* $T\in\mathscr{B}(\mathfrak{H})$ *le modèle fonctionnel dont la fonction caractéristique* $\Theta(\lambda)$ *est donnée par* $\left\{\mathbf{C}, \mathbf{C}, \frac{\sqrt{3}}{2}\lambda^2\right\}$. *Il existe alors un sous-espace* $\mathfrak{H}_1\subset\mathfrak{H}$, *invariant pour* T, *tel que* $\mathfrak{H}_1\neq(S(\mathfrak{H}))^-$ *pour tout* $S\in\{T\}'$.

Démonstration. En effet envisageons la factorisation

$$\frac{1}{2}\sqrt{3}\lambda^2 = \left[\frac{1}{2}\sqrt{3}\lambda \;\; \frac{1}{2}\lambda\right]\cdot\begin{bmatrix}\frac{1}{2}\lambda \\ \frac{1}{2}\sqrt{3}\lambda\end{bmatrix}$$

qui est régulière d'après la prop. VII. 3.5 de [H], et soit $\mathfrak{H}_1 \subset \mathfrak{H}$ le sous-espace invariant, correspondant à cette factorisation. Envisageons un élément $S \in \{T\}'$ fixé. On a

$$S = P_+ Y \Big| \mathfrak{H} = P_+ \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} \Big| \mathfrak{H},$$

$A(\lambda)\colon H^2(\mathbf{C}) \to H^2(\mathbf{C})$ étant une fonction analytique scalaire (dans ce cas $A = A_0$). Supposons que $\mathfrak{H}_1 = (S(\mathfrak{H}))^-$; il existe alors une fonction $\Phi = \begin{bmatrix}\Phi_1 \\ \Phi_2\end{bmatrix}$ avec des fonctions analytiques scalaires $\Phi_i (i = 1, 2)$ telles que $\Phi\Theta = \Theta_1 A_0$ où $\Theta_1 = \begin{bmatrix}\frac{1}{2}\lambda \\ \frac{1}{2}\sqrt{3}\lambda\end{bmatrix}$.

En vertu de la relation $\Theta_1 A_0 = \Phi\Theta$ on a $A = \sqrt{3}\lambda\Phi_1 = \lambda\Phi_2$, d'où $\Phi_2 = \sqrt{3}\Phi_1$. Envisageons un élément $u_0 \in H^2(\mathbf{C})$, $u_0 \neq 0$, et l'élément $\tilde{u}_0 = \begin{bmatrix}-\sqrt{3} \\ 1\end{bmatrix} u_0 \in H^2(\mathfrak{F})$ où, dans ce cas, $\mathfrak{F} = \mathbf{C}^2$. Pour $u', u'' \in H^2(\mathbf{C})$ quelconques, on aura

$$((\Phi u' + \Theta_1 u''), \tilde{u}_0) = \left(\Phi_1 u' + \frac{1}{2}\lambda u'', -u_0\sqrt{3}\right) + \left(\Phi_2 u' + \frac{1}{2}\sqrt{3}\lambda u'', u_0\right) =$$

$$= \left((-\sqrt{3}\Phi_1 + \Phi_2)u', u_0\right) = 0.$$

Donc $\tilde{u}_0 (\neq 0)$ est orthogonal à $(\Phi H^2(\mathbf{C}) + \Theta_1 H^2(\mathbf{C}))^-$ et par conséquent la condition (ii) du théorème n'est pas vérifiée. Donc $\mathfrak{H}_1 \neq (S(\mathfrak{H}))^-$ pour tout $S \in \{T\}'$.

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On a conjecture of Sz.-Nagy and Foiaş

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Sz.-Nagy and Foiaş [10] conjectured that if T is a C_0 contraction with finite multiplicity and $X\in\{T\}'$ then $T|\ker X$ and $(T^*|\ker X^*)^*$ are quasi-similar. Recently, Bercovici [1] and Uchiyama [11] have independently given counter-examples to this conjecture. However, in the present paper we want to establish a weaker form of this conjecture. More specifically, we will show that if T is a $C_0(N)$ contraction and $X\in\{T\}'$ then there exists a $Y\in\{T\}'$ such that $T|\ker X$ and $(T^*|\ker Y^*)^*$ are quasi-similar. Indeed, this follows from the following two main results of Section 1: If T is a $C_0(N)$ contraction, then (1) every invariant subspace for T is of the form $\ker X$ for some $X\in\{T\}'$ (Theorem 1.2) and (2) for every invariant subspace K for T there exists an invariant subspace L for T^* such that $T|K$ is quasi-similar to $(T^*|L)^*$ (Theorem 1.3). In Section 2, we consider the corresponding question for C_{11} contractions. It will be shown that the Sz.-Nagy and Foiaş conjecture holds for completely non-unitary (c.n.u.) C_{11} contractions with finite defect indices. Moreover, result (1) also holds for such operators with bi-invariant subspaces replacing invariant subspaces. ((2) with the same modifications follows from the analogue of the Sz.-Nagy and Foiaş conjecture trivially.) The corresponding results for weak contractions will be considered in Section 3.

In this paper only bounded linear operators on complex, separable Hilbert spaces will be considered. For an operator T, let $\{T\}'$ and $\{T\}''$ denote the commutant and double commutant of T, and let $\operatorname{Lat} T$, $\operatorname{Lat}'' T$ and $\operatorname{Hyperlat} T$ denote the lattices of invariant subspaces, bi-invariant subspaces and hyperinvariant subspaces of T, respectively. If T_1 and T_2 are operators on H_1 and H_2, respectively, $T_1\overset{i}{\prec}T_2$ (resp. $T_1\prec T_2$) denotes that there exists an injection $X\colon H_1\to H_2$ (resp. an injection $X\colon H_1\to H_2$ with dense range, called *quasi-affinity*) which intertwines T_1 and T_2, i.e. $T_1X=XT_2$. T_1 is *quasi-similar* to T_2 ($T_1\sim T_2$) if $T_1\prec T_2$

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and $T_2 \prec T_1$. Readers are referred to Sz.-Nagy and Foiaş [6] for basic definitions and properties of contractions of various classes.

1. $C_0(N)$ contractions. If T is a $C_0(N)$ contraction, then T is quasi-similar to a uniquely determined Jordan operator of the form $S(\varphi_1)\oplus\ldots\oplus S(\varphi_n)$, called the *Jordan model* of T, where $\varphi_1, \ldots, \varphi_n$ are inner functions satisfying $\varphi_j|\varphi_{j-1}$ and $S(\varphi_j)$ denotes the compression of the shift on $H^2\ominus\varphi_j H^2$ for $j=1, 2, \ldots, n$ (cf. [7]). We start with the following lemma.

Lemma 1.1. *Let T be a $C_0(N)$ contraction and let $K\in\operatorname{Lat} T$. Assume that $J=S(\varphi_1)\oplus\ldots\oplus S(\varphi_n)$ and $J_1=S(\psi_1)\oplus\ldots\oplus S(\psi_m)$ are the Jordan models of T and $T|K$, respectively. Then $m\leqq n$ and $\psi_j|\varphi_j$ for $j=1, 2, \ldots, m$.*

Proof. Since $J_1\sim T|K\overset{i}{\prec}T\sim J$, the conclusion follows from [9], Theorem 4.

Theorem 1.2. *Let T be a $C_0(N)$ contraction on H and let K be a subspace of H. Then the following statements are equivalent:*

(1) *$K\in\operatorname{Lat} T$;*

(2) *$K=(\operatorname{Range} X)^-$ for some $X\in\{T\}'$;*

(3) *$K=\ker Y$ for some $Y\in\{T\}'$.*

Proof. It suffices to show (1)⇒(2). Let $J=S(\varphi_1)\oplus\ldots\oplus S(\varphi_n)$ and $J_1=S(\psi_1)\oplus\ldots\oplus S(\psi_m)$ be the Jordan models of T and $T|K$, respectively. Let $V\colon H\to H_1\equiv(H^2\ominus\varphi_1 H^2)\oplus\ldots\oplus(H^2\ominus\varphi_n H^2)$ and $W\colon K_1\equiv(H^2\ominus\psi_1 H^2)\oplus\ldots\oplus(H^2\ominus\psi_m H^2)\to K$ be quasi-affinities intertwining T, J and J_1, $T|K$, respectively. Lemma 1.1 implies that $m\leqq n$ and $\psi_j|\varphi_j$ for $j=1, 2, \ldots, m$; say, $\varphi_j=\psi_j\eta_j$ for each j. Note that $S(\varphi_j)|(\operatorname{Range}\eta_j(S(\varphi_j)))^-\cong S(\psi_j)$ (cf. [9], pp. 315—316). For each j, let Z_j be the operator which implements this unitary equivalence and let $Z\colon H_1\to K_1$ be the operator $Z_1\eta_1(S(\varphi_1))\oplus\ldots\oplus Z_m\eta_m(S(\varphi_m))\oplus\underbrace{0\oplus\ldots\oplus 0}_{n-m}$. Then Z intertwines J, J_1 and has dense range in K_1. Finally, let $X=WZV$. It is obvious that $X\in\{T\}'$ and $K=(\operatorname{Range} X)^-$. This completes the proof.

It is interesting to contrast the preceding theorem with the main result in [14].

Theorem 1.3. *If T is a $C_0(N)$ contraction on H and $K\in\operatorname{Lat} T$, then there exists an $L\in\operatorname{Lat} T^*$ such that $T|K$ is quasi-similar to $(T^*|L)^*$.*

Proof. Let $J=S(\varphi_1)\oplus\ldots\oplus S(\varphi_n)$ and $J_1=S(\psi_1)\oplus\ldots\oplus S(\psi_m)$ be the Jordan models of T and $T|K$, respectively. Then $\tilde{J}\equiv S(\tilde{\varphi}_1)\oplus\ldots\oplus S(\tilde{\varphi}_n)$ is the Jordan model of T^*, where $\tilde{\varphi}_j(z)=\overline{\varphi_j(\bar{z})}$ for each j. Let $V\colon H_1\equiv(H^2\ominus\tilde{\varphi}_1 H^2)\oplus\ldots\oplus(H^2\ominus\tilde{\varphi}_n H^2)\to H$ be the quasi-affinity intertwining $\tilde{J}$ and T^*. From Lemma 1.1 we have $m\leqq n$ and $\psi_j|\varphi_j$ for $j=1, 2, \ldots, m$; say, $\varphi_j=\psi_j\eta_j$ for each j. Then $\tilde{\varphi}_j=\tilde{\psi}_j\tilde{\eta}_j$, which implies that $S(\tilde{\psi}_j)\cong S(\tilde{\varphi}_j)|(\operatorname{Range}\tilde{\eta}_j(S(\tilde{\varphi}_j)))^-$ for $j=1, \ldots, m$.

For each j, let Z_j be the operator which implements this unitary equivalence and let $Z=Z_1\oplus\ldots\oplus Z_m$. Let $\tilde{J}_1=S(\tilde{\psi}_1)\oplus\ldots\oplus S(\tilde{\psi}_m)$ on $\tilde{K}_1\equiv(H^2\ominus\tilde{\psi}_1H^2)\oplus\ldots\oplus(H^2\ominus\tilde{\psi}_mH^2)$ and let $L=(VZ\tilde{K}_1)^-$. Then $L\in\operatorname{Lat}T^*$ and $\tilde{J}_1\prec T^*|L$. It follows that $\tilde{J}_1\sim T^*|L$ and hence $(T^*|L)^*\sim\tilde{J}_1^*\cong S(\psi_1)\oplus\ldots\oplus S(\psi_m)\sim T|K$, completing the proof.

Corollary 1.4. *If T is a $C_0(N)$ contraction and $X\in\{T\}'$, then there exists another $Y\in\{T\}'$ such that $T|\ker X$ is quasi-similar to $(T^*|\ker Y^*)^*$.*

Note that Theorems 1.2 and 1.3 generalize the corresponding results for operators on finite-dimensional spaces proved by HALMOS (cf. [4], Theorems 2 and 3).*)

2. C_{11} contractions. Let T be a c.n.u. C_{11} contraction with finite defect indices defined on $H\equiv\langle H_n^2\oplus\overline{\Delta L_n^2}\rangle\ominus\{\Theta_Tw\oplus\Delta w\colon w\in H_n^2\}$ by $T(f\oplus g)=P(e^{it}f\oplus e^{it}g)$ for $f\oplus g\in H$, where $\Delta=(1-\Theta_T^*\Theta_T)^{1/2}$ and P denotes the (orthogonal) projection onto H. Let U be the operator of multiplication by e^{it} on $(\Delta_*L_n^2)^-$, where $\Delta_*=(1-\Theta_T\Theta_T^*)^{1/2}$, and let $X\colon H\to(\Delta_*L_n^2)^-$ be the quasi-affinity $X(f\oplus g)=-\Delta_*f+\Theta_Tg$ which intertwines T and U (cf. [18], Lemma 3.4). Let δ be an outer scaler multiple of Θ_T and let Ω be a contractive analytic function such that $\Omega\Theta_T=\Theta_T\Omega=\delta 1$. Note that we have $\Omega\Delta_*=\Delta\Omega$.

Define the operator $Y\colon(\Delta_*L_n^2)^-\to H$ by $Yu=P(0\oplus\Omega u)$ for $u\in(\Delta_*L_n^2)^-$. Note that Ωu is in $(\Delta L_n^2)^-$ for any $u\in(\Delta_*L_n^2)^-$. Indeed, if $u\in(\Delta_*L_n^2)^-$, there exists a sequence of vectors $\{u_m\}$ in L_n^2 such that $\Delta_*u_m\to u$ in norm as $m\to\infty$. Since $\Omega\Delta_*u_m=\Delta\Omega u_m\in\Delta L_n^2$ for all m and $\Omega\Delta_*u_m\to\Omega u$, we conclude that $\Omega u\in(\Delta L_n^2)^-$ as asserted.

Lemma 2.1. *Let T, U, X and Y be as above. Then* (1) $YX=\delta(T)$ *and* $XY=\delta(U)$; (2) Y *is a quasi-affinity intertwining* U *and* T.

Proof. (1) For any $P(0\oplus g)\in H$, we have

$$YX(P(0\oplus g)) = YX((0\oplus g)-(\Theta_Tw\oplus\Delta w)) = Y(-\Delta_*(-\Theta_Tw)+\Theta_T(g-\Delta w)) =$$

$$= Y(\Theta_Tg) = P(0\oplus\Omega\Theta_Tg) = P(0\oplus\delta g) = \delta(T)P(0\oplus g),$$

where $w\in H_n^2$ and in the third equality we used the relation $\Delta_*\Theta_T=\Theta_T\Delta$. Since $\{P(0\oplus g)\colon g\in(\Delta L_n^2)^-\}$ is dense in H (cf. [19], proof of Lemma 2), we conclude that

*) *Editor's Note:* The results in this section are actually true for arbitrary C_0 contractions on (not necessarily separable) Hilbert spacer. We have only to refer to the existence of the Jordan model and the validity of Lemma 1.1 in the general case. These have been proved by H. BERCOVICI (On the Jordan model of C_0 operators. II, *Acta Sci. Math.*, **42** (1980), 43—56). He also proved Theorem 1.2 for arbitrary C_0 contractions.

$YX=\delta(T)$. In a similar fashion, for any $u\in(\Delta_* L^2_n)^-$,

$$XYu = XP(0\oplus\Omega u) = \Theta_T\Omega u = \delta u = \delta(U)u.$$

Hence $XY=\delta(U)$.

(2) For any $u\in(\Delta_* L^2_n)^-$,

$$YUu = Y(e^{it}u) = P(0\oplus\Omega e^{it}u) = TP(0\oplus\Omega u) = TYu.$$

This shows that Y intertwines U and T. That Y is a quasi-affinity follows from (1) and the fact that both $\delta(T)$ and $\delta(U)$ are quasi-affinities (since $\delta\not\equiv 0$ is an outer function; cf. [6], pp. 118 and 121, resp.).

To prove the analogue of the Sz.-Nagy and Foiaş conjecture for C_{11} contractions, we need the next theorem. It implies that for c.n.u. C_{11} contractions with finite defect indices, the lattices of invariant subspaces, bi-invariant subspaces and hyperinvariant subspaces are all preserved under quasi-similarities. Note that the assertions concerning bi-invariant and hyperinvariant subspaces have been obtained before by more elaborate methods (cf. [18] and [15]). By introducing the operator Y we are able to prove all three assertions in one stroke.

Theorem 2.2. *Let T, U, X and Y be as above. Then* $\operatorname{Lat} T\cong\operatorname{Lat} U$, $\operatorname{Lat}'' T\cong\operatorname{Lat}'' U$ *and* $\operatorname{Hyperlat} T\cong\operatorname{Hyperlat} U$. *Moreover, in each case the (lattice) isomorphisms are implemented by the mappings $K\to\overline{XK}$ and $L\to\overline{YL}$ where K is in* $\operatorname{Lat} T$, $\operatorname{Lat}'' T$ *or* $\operatorname{Hyperlat} T$ *and L in* $\operatorname{Lat} U$, $\operatorname{Lat}'' U$ *or* $\operatorname{Hyperlat} U$, *and $T|K$ is quasi-similar to $U|\overline{XK}$.*

We first prove the following lemma.

Lemma 2.3. *Let U be an absolutely continuous unitary operator and let $L\in\operatorname{Lat} U$. If δ is an outer function, then $\delta(U|L)$ is a quasi-affinity on L.*

Proof. Since $U'=U|L$ is a contraction, we may consider the canonical decomposition $U'=U_1\oplus U_2$ of U', where U_1 and U_2 are the unitary and c.n.u. parts of U', respectively. Then $\delta(U')=\delta(U_1)\oplus\delta(U_2)$. That $\delta\not\equiv 0$ is an outer function implies that both $\delta(U_1)$ and $\delta(U_2)$ are quasi-affinities (cf. [6], pp. 121 and 118, resp.). It follows that $\delta(U')$ is a quasi-affinity.

Proof of Theorem 2.2. Since X and Y intertwine T and U, it is easily seen that $\overline{XK}\in\operatorname{Lat} U$ and $\overline{YL}\in\operatorname{Lat} T$ for any $K\in\operatorname{Lat} T$ and $L\in\operatorname{Lat} U$. Moreover, $(Y\overline{XK})^-=(YXK)^-=(\delta(T)K)^-=(\delta(T|K)K)^-=K$ and $(X\overline{YL})^-=(XYL)^-=(\delta(U)L)^-=(\delta(U|L)L)^-=L$ by Lemmas 2.1 and 2.3. We infer that the mappings $K\to\overline{XK}$ and $L\to\overline{YL}$ implement the (lattice) isomorphisms between $\operatorname{Lat} T$ and $\operatorname{Lat} U$ and $T|K$ is quasi-similar to $U|\overline{XK}$.

To complete the proof it suffices to show that for any $K\in \mathrm{Lat}'' T$ or Hyperlat T, $\overline{XK}\in \mathrm{Lat}'' U$ or Hyperlat U and for any $L\in \mathrm{Lat}'' U$ or Hyperlat U, $\overline{YL}\in \mathrm{Lat}'' T$ or Hyperlat T, respectively. If $K\in \mathrm{Lat}'' T$, then $\sigma(T|K)\subseteq\sigma(T)$ (cf. [20], Theorem 3), and so $T|K\in C_{11}$. Therefore $T|K$ is quasi-similar to a unitary operator. Since $T|K\prec U|\overline{XK}$, we infer by Lemma 4.1 of [3] that $\overline{XK}$ is a reducing subspace of U, and so $\overline{XK}\in \mathrm{Lat}'' U$. On the other hand, if $L\in \mathrm{Lat}'' U$, then $\overline{YL}=\{P(0\oplus\oplus\Omega u)\colon u\in L\}^-$. An operator S in $\{T\}''$ must be of the form $P\begin{bmatrix} A & O\\ B & C\end{bmatrix}$, where A is a bounded analytic function, B and C are bounded measurable functions and C is scalar-valued satisfying $A\Theta_T=\Theta_T A_0$ and $B\Theta_T+C\Delta=\Delta A_0$ for some bounded analytic function A_0 (cf. [19], Lemma 2). Hence $S\big(P(0\oplus\Omega u)\big)=P\begin{bmatrix} A & O\\ B & C\end{bmatrix}P\begin{bmatrix} O\\ \Omega u\end{bmatrix}= =P\begin{bmatrix} O\\ C\Omega u\end{bmatrix}=P\begin{bmatrix} O\\ \Omega C u\end{bmatrix}\in\overline{YL}$ for any $u\in L$, since $L\in \mathrm{Lat}'' U$ and $C\in\{U\}''$. It follows that $S\overline{YL}\subseteq\overline{YL}$, whence $\overline{YL}\in \mathrm{Lat}'' T$.

If $K\in$ Hyperlat T, then $\overline{XK}\in$ Hyperlat U by [15], Corollary 1. Now let $L\in$ Hyperlat U and let $S=P\begin{bmatrix} A & 0\\ B & C\end{bmatrix}$ be an operator in $\{T\}'$, where A, B and C are as above except that C may not be scalar-valued (cf. [8]). As before, $\overline{YL}=\{P(0\oplus\Omega u)\colon u\in L\}^-$ and $SP(0\oplus\Omega u)=P\begin{bmatrix} 0\\ C\Omega u\end{bmatrix}$ for any $u\in L$. Note that L, being hyperinvariant for U, is of the form $\chi_F(\Delta_* L_n^2)^-$ for some Borel subset F of the unit circle. Assume that $u=\chi_F\Delta_* v$ for some $v\in L_n^2$. Then $C\Omega u=\chi_F C\Omega\Delta_* v$. Note that $A\Theta_T=\Theta_T A_0$ and $B\Theta_T+C\Delta=\Delta A_0$ imply that $\Omega A=A_0\Omega$ and $B\delta+ +C\Delta\Omega=\Delta A_0\Omega$. Thus

$$C\Omega\Delta_* = C\Delta\Omega = \Delta A_0\Omega-B\delta = \Delta\Omega A-\Omega\Theta_T B = \Omega(\Delta_* A-\Theta_T B)$$

and we have $C\Omega u=\chi_F\Omega(\Delta_* A-\Theta_T B)v$. Note that $\Theta_T Bv\in(\Delta_* L_n^2)^-$ and hence $C\Omega u=\Omega w$, where $w=\chi_F(\Delta_* A-\Theta_T B)v\in L$. This shows that $SP(0\oplus\Omega u)\in\overline{YL}$ for any $u=\chi_F\Delta_* v\in L$. Since $\{\chi_F\Delta_* v\colon v\in L_n^2\}$ is dense in L, we conclude that $SP(0\oplus\Omega u)\in\overline{YL}$ for all $u\in L$. Hence $S\overline{YL}\subseteq\overline{YL}$ and $\overline{YL}\in$ Hyperlat T, completing the proof.

The next theorem is the analogue of Theorem 1.2 for C_{11} contractions. It should be contrasted with [16], Theorem 3.6.

Theorem 2.4. *Let T be a c.n.u. C_{11} contraction on H with finite defect indices and let $K\subseteq H$ be a subspace. Then the following statements are equivalent:*

(1) $K\in \mathrm{Lat}'' T$;

(2) $K=(\mathrm{Range}\ S)^-$ *for some* $S\in\{T\}'$;

(3) $K=\ker V$ *for some* $V\in\{T\}'$.

Proof. It suffices to show (1)$\Rightarrow$(2). Let U, X and Y be defined as before and $L=\overline{XK}$. Then L, being in $\mathrm{Lat}''\,U$, reduces U. Therefore $L=(W(\Delta_* L^2_n)^-)^-$ for some $W\in\{U\}'$. (W may be taken to be the (orthogonal) projection from $(\Delta_* L^2_n)^-$ onto L.) Let $S=YWX$. Then $S\in\{T\}'$ and $(\mathrm{Range}\, S)^-=(YWXH)^-=$ $=(YW(\Delta_* L^2_n)^-)^-=(YL)^-=(YXK)^-=K$ by Theorem 2.2.

In the remainder of this section we will show that the Sz.-Nagy and Foiaş conjecture holds for C_{11} contractions.

Lemma 2.5. *If T is a normal operator on H with finite multiplicity and $X\in\{T\}'$, then $T|\ker X$ is unitarily equivalent to $(T^*|\ker X^*)^*$.*

Recall that the *multiplicity* of an operator T on H is the least cardinal number of a set of vectors in H which, together with their transforms by $T, T^2, \ldots$, span H.

Proof. Note that $(\mathrm{Range}\, X)^-$ and $(\mathrm{Range}\, X^*)^-$ reduce T and $T|(\mathrm{Range}\, X)^-$ is unitarily equivalent to $T|(\mathrm{Range}\, X^*)^-$ (cf. [3], Lemma 4.1). Let $\mathscr{A}$ be the von Neumann algebra generated by T and I. Since T is normal, $\mathscr{A}$ is abelian, hence finite. On the other hand, T has finite multiplicity implies that $\mathscr{A}'$ is also finite. By [5], Theorem 3, we conclude that $T|\ker X$ is unitarily equivalent to $T|\ker X^*=(T^*|\ker X^*)^*$.

Theorem 2.6. *If T is a c.n.u. C_{11} contraction on H with finite defect indices and $X\in\{T\}'$, then $T|\ker X$ is quasi-similar to $(T^*|\ker X^*)^*$.*

Proof. Let U be the operator of multiplication by e^{it} on $(\Delta_* L^2_n)^-$ and let $Z\colon H\to(\Delta_* L^2_n)^-$ and $Y\colon(\Delta_* L^2_n)^-\to H$ be the quasi-affinities defined in the beginning of Section 2 such that $YZ=\delta(T)$ and $ZY=\delta(U)$, where δ is some outer function. It is easily seen that $ZXY\in\{U\}'$, $(Y(\ker ZXY))^-\subseteq\ker X$ and $(Z(\ker X))^-\subseteq$ $\subseteq\ker ZXY$. From Theorem 2.2 we infer that $\ker ZXY=(ZY(\ker ZXY))^-\subseteq$ $\subseteq(Z(\ker X))^-\subseteq\ker ZXY$. Hence $(Z(\ker X))^-=\ker ZXY$ and $T|\ker X$ is quasi-similar to $U|\ker ZXY$. Note that Z^* and Y^* are also quasi-affinities satisfying $Z^*Y^*=\tilde{\delta}(T^*)$ and $Y^*Z^*=\tilde{\delta}(U^*)$ where $\tilde{\delta}(z)=\overline{\delta(\bar{z})}$ is outer. A similar argument as above shows that $T^*|\ker X^*$ is quasi-similar to $U^*|\ker(ZXY)^*$. Lemma 2.5 says that $U|\ker ZXY$ is unitarily equivalent to $U|\ker(ZXY)^*$. We conclude that $T|\ker X$ is quasi-similar to $(T^*|\ker X^*)^*$ as asserted.

3. Weak contractions. In this section we generalize some results in Sections 1 and 2 to weak contractions. The next theorem is the generalization of Theorems 1.2 and 2.4. It should be contrasted with [17], Theorem 3.8 and Corollary 3.9.

Theorem 3.1. *Let T be a c.n.u. weak contraction on H with finite defect indices and let $K\subseteq H$ be a subspace. Then the following statements are equivalent:*

(1) $K\in\mathrm{Lat}''\,T$;

(2) $K=(\operatorname{Range} S)^-$ *for some* $S\in\{T\}'$;
(3) $K=\ker V$ *for some* $V\in\{T\}'$.

Proof. We have only to prove (1)⇒(2). Let $H_0, H_1\subseteq H$ be the invariant subspaces of T on which the C_0 and C_{11} parts of T act, respectively. Since $K\in\operatorname{Lat}'' T$, $T|K$ is also a weak contraction (cf. [18], Theorem 4.1). Hence we may also consider the subspaces $K_0, K_1\subseteq K$ on which the C_0 and C_{11} parts of $T|K$ act. Since $K_0\in\operatorname{Lat}'' T_0$ and $K_1\in\operatorname{Lat}'' T_1$ (cf. [18], Theorem 4.1), $K_0=\overline{S_0H_0}$ and $K_1=\overline{S_1H_1}$ for some $S_0\in\{T_0\}'$ and $S_1\in\{T_1\}'$ by Theorems 1.2 and 2.4. We also have $H_0=\overline{V_0H}$ and $H_1=\overline{V_1H}$ for some $V_0, V_1\in\{T\}''$ (cf. [17], Theorem 3.1 and [6], p. 334, resp.). Let $S=S_0V_0+S_1V_1$. It is easily seen that $S\in\{T\}'$ and $(\operatorname{Range} S)^-=(S_0V_0H+S_1V_1H)^-=(S_0H_0+S_1H_1)^-=K_0\vee K_1=K$. This completes the proof.

The next theorem (partially) generalizes Theorems 1.3 and 2.6.

Theorem 3.2. *If T is a c.n.u. weak contraction on H with finite defect indices and $K\in\operatorname{Lat}'' T$, then there exists an $L\in\operatorname{Lat}'' T^*$ such that $T|K$ is quasi-similar to $(T^*|L)^*$.*

Proof. As in the proof of the preceding theorem, let $H_0, H_1\subseteq H$ and $K_0, K_1\subseteq K$ be the invariant subspaces for T, $T|K$ such that $T_0\equiv T|H_0$ and $T_1\equiv T|H_1$ are the C_0 and C_{11} parts of T and $T|K_0$ and $T|K_1$ are the C_0 and C_{11} parts of $T|K$, respectively. Since $K_0\in\operatorname{Lat}'' T_0$ and $K_1\in\operatorname{Lat}'' T_1$, there exist $L_0\in\operatorname{Lat}'' T_0^*$ and $L_1\in\operatorname{Lat}'' T_1^*$ such that $T_0|K_0\sim(T_0^*|L_0)^*$ and $T_1|K_1\sim(T_1^*|L_1)^*$ by Theorems 1.3 and 2.6. Let $L'=L_0\oplus L_1\in\operatorname{Lat}'' T_0^*\oplus\operatorname{Lat}'' T_1^*=\operatorname{Lat}''(T_0^*\oplus T_1^*)$ (cf. [2], Prop. 1.3 and Lemma 4.4). By [12], Theorem 1, $T|K\sim(T_0|K_0)\oplus(T_1|K_1)\sim$ $\sim(T_0^*|L_0)^*\oplus(T_1^*)|L_1)^* = \big((T_0^*\oplus T_1^*)|L'\big)^*$. Similarly, $T\sim T_0\oplus T_1$, whence $T^*\sim$ $\sim T_0^*\oplus T_1^*$. Note that quasi-similar c.n.u. weak contractions with finite defect indices have isomorphic bi-invariant subspace lattices and the restrictions of the weak contractions to the corresponding bi-invariant subspaces are quasi-similar to each other (cf. [18], Added in proof). We infer that there is an $L\in\operatorname{Lat}'' T^*$ such that $T^*|L\sim(T_0^*\oplus T_1^*)|L'$. Hence $T|K\sim(T^*|L)^*$ as asserted.

We conclude this paper by a simple observation that if T is a weak contraction and $X\in\{T\}'$ then $T|(\operatorname{Range} X)^-$ is quasi-similar to $\big(T^*|(\operatorname{Range} X^*)^-\big)^*$. Indeed, this follows from the following

Lemma 3.3. *If T is an operator on H and $X\in\{T\}'$, then $\big(T^*|(\operatorname{Range} X^*)^-\big)^*\prec$ $\prec T|(\operatorname{Range} X)^-$.*

Proof. Let X_1 be the operator $X|(\operatorname{Range} X^*)^-$ from $(\operatorname{Range} X^*)^-$ to $(\operatorname{Range} X)^-$. It is routine to check that X_1 is a quasi-affinity intertwining $\big(T^*|(\operatorname{Range} X^*)^-\big)^*$ and $T|(\operatorname{Range} X)^-$. We leave the details to the readers.

Corollary 3.4. *If T is a C_0 contraction, a C_{11} contraction or a weak contraction and $X \in \{T\}'$, then $T|(\mathrm{Range}\, X)^-$ is quasi-similar to $(T^*|(\mathrm{Range}\, X^*)^-)^*$.*

Proof. This follows from Lemma 3.3 and [13], Lemma 3.

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Bibliographie

R. Azencott—Y. Guivarc'h—R. F. Gundy, Ecole d'Eté de Probabilités de Saint-Fluor (VIII—1978, P. L. Hennequin). Ed. (Lecture Notes in Mathematics, 774), XIII+334 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1980.

The volume containes three long survey articles. Azencott describes the recent flourishment of large deviation theory and its applications rather thoroughly, save the definitive paper by Groeneboom, Oosterhoff and Ruymgaart [*Ann. Probability*, 7 (1979), 553—586]. Guivarc'h investigates some asymptotic properties of random matrices excluding limit distribution results. Finally, Gundy overviews martingale inequalities (also for doubly indiced sequences) and the beautiful results on the intimate connection with H^p spaces.

Sándor Csörgő (Szeged)

L. W. Beineke and R. J. Wilson, Selected Topics in Graph Theory, XII+451 pages, Academic Press, London—New York—San Francisco 1978.

The literature of graph theory has grown at an enormous speed in the fast 10 years both in research papers and textbooks. Accordingly, there are many textbooks treating the basic results; these, however, have to remain on the surface and cannot go into the discussion of the recent result. There are some research monographs around, but these cover very small portion of the field. Many people who already know the basics would like to have an overview of what is going on in graph theory, what are the most important new results, what are the topics currently attracting the most interest, and what are the most exciting unsolved problems. The book reviewed here answers such questions. It is a collection of surveys wrote by outstanding researchers in the field. The advantage over, say, a conference proceedings with survey papers is that the topics are carefully chosen and the style, notation and terminology of the essays are unified by the editors. It contains 14 surveys (A. T. White—L. W. Beineke: Topological graph theory, A. T. White: The proof of the Heawood conjecture, D. R. Woodall—R. J. Wilson: The Appel-Haken proof of the Four-Color Theorem, S. Fiorini—R. J. Wilson: Edge colorings of graphs, J. C. Bermond: Hamiltonian graphs, K. B. Reid—L. W. Beineke: Tournaments, C. St. J. A. Nash—Willams: The reconstruction problem, D. R. Woodall: Minimax theorems in graph theory, R. L. Hemminger and L. W. Beineke: Line graphs and line digraphs, P. J. Cameron: Strongly regular graphs, T. D. Parsons: Ramsey graph theory, E. M. Palmer: The enumeration of graphs, R. C. Reid: Some applications of computers in graph theory).

It is natural that no such selection can cover all topics which are currently in the main stream of research, and there would be as many different selections as researchers. But certainly these essays give a very good insight into those trends which they treat and some of them in fact yield new and exciting points of view. Hopefully further topics will be covered by a continuation of this volume and this standard will be maintained.

László Lovász (Szeged)

D. van Dalen, Logic and Structures, VII+172 pages, Springer-Verlag, Berlin, Heidelberg, New York, 1980.

"This book provides an efficient introduction to logic for students of mathematics". It does really so.

Chapter 1 is devoted to propositional logic. The notions and basic properties of logical connectives etc. are established heuristically and only then the rigorous development follows. This method makes it easier to understand the aims of the theory for beginners. The material is arranged in a way similar to that used in the second part at predicate logic and this is a great help when reading the second part of the book. Both semantics and formal deductions (Gentzen's System) as well as the bridge between them: the Completeness Theorem are treated. Unfortunately the formal proofs are not easy to read and this might spoil one's interest in the subject. This is, however, by the nature of formal logic and not the fault of the book; also, the author often leaves the formal proofs to the reader.

The second chapter develops the basic facts about languages and structures, and illustrates the results through examples. It carefully points out those delicate steps (e. g. the use of "=" in the given language and in the meta-language) which usually cause problems for the beginners. Here, also, it is a hard work to follow the formal proofs.

In the last part of the book the author proves the completeness theorem (using Henkin constants) together with many of its consequences: compactness theorem, Skolem-Löwenheim theorems, axiomatizability, etc. and continues with the elements of model theory (elementary substructures, diagram language, Skolem functions). The results are also illustrated by such examples as the non-standard models of arithmetic and of the reals or a non-standard proof of König's lemma.

Finally a very brief description is given about second order logic.

The book contains a number of easier or harder exercises which help to practice and master the material.

V. Totik (Szeged)

M. Denker and K. Jacobs, Eds., Ergodic Theory, Proceedings, Oberwolfach, Germany 1978 (Lecture Notes in Mathematics, 729), XII+209 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1979.

The twenty-one research papers of this collection, read at an Oberwolfach conference, June 11—17, 1978, present new results in many directions (ergodicity categories, topological dynamics, L_p spaces, measure preserving homeomorphisms, reparametrization, normalizer groups, partitions and Rochlin sets, maximal and invariant measures, topological entropy, weak-mixing Markov operator semi-groups, ergodic group automorphisms, skew products, balancing averages, code lengths, the Lorentz attractor). All these topics still seem to belong to one branch of mathematics, namely to ergodic theory. The volume is dedicated to the memory of Rufus Bowen.

Sándor Csörgő (Szeged)

A. Dold and B. Eckham, Combinatorial Mathematics, Proceedings, Armidale, Australia, 1978. (Lecture Notes in Mathematics 748) IX+206 pp. Springer-Verlag, Berlin—Heidelberg—New York, 1979.

The book contains 18 papers delivered at the Sixth Australian Conference on Combinatorial Mathematics, representing mainly the fast developing Australian School of combinatorics. The three invited survey papers included in the volume are: R. B. Eggleton—D. A. Holton: Graphic sequences; Sheila Oakes Macdonald: Combinatorics — a branch of group theory? and B. D. McKay—R. G. Stanton: The current status of the generalized Moore graph problem.

L. Lovász (Szeged)

Th. Gasser and M. Rosenblatt, Eds., Smoothing Techniques for Curve Estimation, Proceedings, Heidelberg 1979 (Lecture Notes in Mathematics, 757), 245 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1979.

One of the basic problems of theoretical and applied statistics is: how to smooth the data to form an estimator (a sequence of stochastic processes) for the theoretical curve (distribution, quantile, or denisty function, or the derivatives of the latter, the regression function), or some functional of it (mode, multiple regression, etc.) to be estimated? The twelve research or survey papers of this collection, presented at a Heidelberg Workshop, April 2—4, 1979, offer a variety of such techniques for nonparametric curve estimation. Most of the papers deal with asymptotic properties of kernel, nearest neighbor, least squares, spline-function, quantile and M-estimators, and of robust and Tukey smoothers.

Sándor Csörgő (Szeged)

George Grätzer, General Lattice Theory (Lehrbücher und Monographien aus dem Gebiete der exacten Wissenchaften, mathematische Reihe, Band 52), XIII+381 pages, Birkhäuser Verlag, Basel und Stuttgart, 1978.

This is a basic monograph on lattice theory, which may serve as a textbook of the subject as well as a reference book for researchers working in the area. We have every reason to predict that this book will play the same fundamental role in lattice theory that the same author's "Universal Algebra" has played for a decade now in the theory of general algebraic systems.

The author's view is that in the seventies it is impossible to cover all of lattice theory in a book, thus he omits a great deal (ordered algebraic systems being one example), discusses only the basics, but the discussion goes into considerable depth. To carry out even such a project is a very difficult task at an age when the development is so rapid that no year passes without a remarkable breakthrough in one or another branch of lattice theory. Such areas as the theory of transferability, which is one of the author's major research interests, and the theory of pseudocomplemented distributive lattices, a topic thoroughly dealt with in Grätzer's earlier book "Lattice Theory: First Concepts and Distributive Lattices" (later on referred to as FC) have to be omitted or only briefly mentioned in order to reduce the material and to keep the size of the book within reasonable limits. But the result makes up for these sacrifices: the remaining material reflectes all what is dealt with in present day's lattice theory and the depth of the treatise makes the book an excellent course to bring the student from the very beginning to a point where he can start researches on his own.

The author breaks with the conception of building up lattice theory proceeding from partially ordered sets through lattices and modular lattices to distributive lattices. Distributive lattices are treated as a first priority in the book. This is justified by historical reasons (lattice theory started with distributive lattices) as well as by the fact that in the applications disrtibutive lattices play the most essential part. This approach has the additional advantages that, later on in the book, distributive lattices can serve as a model for all of lattice theory and the reader can reach deep results early. After an introductory chapter (Chapter I. First Concepts), where, among other things, free lattices, partial lattices, and finitely presented lattices are considered, Chapter II. deals with distributive lattices. These two chapters reproduce most of the material of FC, thus the book is not the companion volume of FC that was announced there, but a self-contained work.

Some highlights of the book: Chapter III. Among other things, Grätzer's and Schmidt's characterization of lattices with Boolean congruence lattice is presented. Chapter IV. A thorough discussion of type 2 and type 3 representability and an account on Jónsson's Arguesian equation is given. Chapter V. contains Baker's method of constructing equational bases, Herrmann's proof of McKenzie's finite basis theorem, the equational characterization of the variety M_3 and a discussion of the Amalgamation Property. Chapter VI. describes the structure of free lattices, $\mathscr{C}$-reduced free products, and gives a proof of Dilworth's famous theorem on the embeddability of a lattice into a uniquely complemented one. To present all these results and so many more in a book of this size needed substantial simplifications of the original proofs. Reading the book, one has the feeling that the proofs presented are the simplest possible.

The reader of this review might be interested in the table of contents: I. First Concepts: Two Definitions of Lattices; How to Describe Lattices; Some Algebraic Concepts; Polynomials, Identities, and Inequalities; Free Lattices; Special Elements. II. Distributive Lattices: Characterization Theorems and Representation Theorems; Polynomials and Freeness; Congruence Relations; Boolean Algebras R-generated by a Distributive Lattice; Topological Representations; Distributive Lattices with Pseudocomplementation. III. Congruences and Ideals: Weak Projectivity and Congruences; Distributive, Standard, and Neutral Elements; Distributive, Standard, and Neutral Ideals; Structure Theorems. IV. Modular and Semimodular Lattices: Modular Lattices; Semimodular Lattices; Geometric Lattices; Partition Lattices; Complemented Modular Lattices. V. Equational Classes of Lattices: Characterizations of Equational Classes; The Lattice of Equational Classes of Lattices; Finding Equational Bases; The Amalgamation Property. VI. Free Products: Free Products of Lattices; The Structure of Free Lattices; Reduced Free Products; Hopfian Lattices. Concluding Remarks. Bibliography. Table of Notation. Index

Each chapter is completed by a section "Further Topics and References", which is designed to bring the reader up-to-date in the most recent development. There is also a list of unsolved problems. The bibliography consists of more than 700 items. The short "Concluding Remarks" refer to some of the most important papers in the congruence representation theory and in the theory of transferability. It also contains hints to some relevant books and papers on the theory of Riesz spaces (vector lattices) and on orthomodular lattices. This part reflects the connections of lattice theory with analysis and physics.

Anybody doing research in or close to lattice theory is advised to read this book.

A. P. Huhn (Szeged)

C. C. Heyde—E. Seneta, I. J. Bienaymé. Statistical Theory Anticipated (Studies in the History of Mathematics and Phisical Sciences, 3), XIV+172 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1977.

Most of us knew of Bienaymé only that has found the Chebyshev inequality some time (14 years, as it happens) before Chebyshev. The discovery of Heyde and Seneta [*Biometrika*, 59 (1972), 406—409] that he knew the key critacality theorem for the simple ("Galton-Watson") branching process in a correct form anticipating Galton and Watson by some 28 years (and the first formerly known correct statement of it by 85 years) has already suggested that there was much more in the man. Now here is a most enjoyable account on his scientific activities and life, and we can see a great, almost entirely forgotten probabilist and statistician of the 19th century in his full richness. The chapter headings (Hisorical background; Demography and social sciences; Homogeneity and stability of statiscical trials; Linear least squares; Other probability and statistics; Miscellaneous writings) can indicate his wide scope, and reading this excellent book justifies its title. But there is more in this monograph. As the authors write "the evolution of probability and statistics is fairly well documented up to the time of Laplace, and the developments of the twentieth century are widely appreciated. The intervening period of the last three quarters of the nineteenth century is the least well-understood period in the history of the subject". No doubt, this widely documented book clears up the mystery a great deal.

Sándor Csörgő (Szeged)

H. Heyer, Ed., Probability Measures on Groups, Proceedings, Oberwolfach, Germany 1978. (Lecture Notes in Mathematics, 706), XIII+348 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1979.

This is a collection of twenty-five research papers presented at an Oberwolfach conference January 29 — February 4, 1978. The editor roughly classifies these papers as belonging to one of the following five main topics: 1. Infinite convolutions of probability measures on groups and semigroups; 2. Continuous semigroups; 3. Special classes of probability measures (infinitely divisible and stable measures on groups); 4. Random walks on groups and homogeneous spaces (potential theory, noncommutative renewal theory, local limit theorems); 5. Group representations and probability.

Sándor Csörgő (Szeged)

I. A. Ibragimov—Y. A. Rozanov, Gaussian Random Processes (Applications of Mathematics, 9) X+275 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1978.

This monograph (a translation of the 1970 Russian original) provides detailed investigations into three quite narrow but otherwise important problems concerning Gaussian stationary processes. The first is the equivalence problem of determining the conditions for mutual absolute continuity of Gaussian stationary measures, and to obtain the corresponding densities. The second problem is to describe the classes of spectral measures corresponding to regular strong mixing stationary processes. For the Gaussian case the monograph offers the complete solution. These are the strongest parts of the book, being the standard reference in this respect. The third problem is the estimation of the unknown mean of a stationary Gaussian process. This part of the monograph has perhaps been superseded in the meantime.

Other important topics (e. g. sample path properties,) are not discussed, hence the title of the monograph sounds more general than justified by the content.

Sándor Csörgő (Szeged)

J. G. Kalbfleisch, Probability and Statistical Inference I, II (Universitext), 342, 316 pages, with 35 illustrations, Springer-Verlag, New York—Heidelberg—Berlin.

These two volumes constitute a well-compiled fair introductory textbook for beginners knowing only freshman calculus. The sixteen chapter headings (Introduction; Equi-probable outcomes; The calculus of probability; Discrete variates; Mean and variance; Continuous variates; Bivariate continuous distributions; Generating functions; Likelihood methods, Two-parameter likelihoods; Checking the model; Test of significance; Intervals from significance test, Inferences for normal distribution parameters; Fitting a straight line; Topics in statistical inference) describe the scope and contents. The author's emphasis is on "applications and logical principles rather than mathematical theory". The text is, consequently, full with elementary examples. Each chapter ends with "review problems", solutions to some of which are given at the end of both volumes. The style is clear and very British. (The present reviewer likes it.)

Sándor Csörgő (Szeged)

Sofya Kovalevskaya: A Russian Childhood (Translated and introduced by Beatrice Stillman), Springer-Verlag, New York, etc. 1978.

Sofya Kovalevskaya (1850—1891), the first female professor of mathematics (Stockholm University) and first female corresponding member of the Russian Academy of Sciences, was not only an outstanding mathematician but a highly gifted writer as well. Her memoir gives a vivid description of the country life of the family of Sofya's father (who, apparently, was a descendant of Matthias Corvinus, king of Hungary in the 15^{th} century). Some chapters, especially the fourth, reach truly high aesthetic level. The last two chapters (the relation of the Kovalevskaya sisters to F. M. Dostoevsky) are of special interest.

Ms. Stillman contributed very much to the book by her notes as well. (However, according to p. 250, in the "Notes on the General Literatute", about items of what she calls the secondary literature, she does not seem to know about a popular Hungarian "Mädchenroman" on "Professor Sonya".)

András Recski (Budapest)

Kenneth A. Ross, Elementary Analysis: The Theory of Calculus, VI+264 pp., Springer-Verlag, New York, Heidelberg, Berlin 1980.

"Designed for students having no previous experience with rigorous proofs, this text on analysis can be used immediately following standard calculus courses". This book is an excellent bridge between elementary calculus and more advanced studies in real analysis.

Chapter I contains the properties of the natural numbers and ordered fields.

Chapter II is devoted to a thorough study of sequences and series. The material is arranged in a usual way, the only exception is perhaps the Bolzano—Weierstrass Theorem: this is derived directly from the completeness axiom without the "halving procedure". The author strives to use only the necessary concepts and probably this effort made him avoid the concept of a closed set (treats it only in the optional § 13). However, the notion "closed" is so important that it would be worth while to use it even in an elementary book.

In Chapters III and IV the basic properties of continuous functions and function series are discussed. Chapter IV may be considered also as a very brief introduction to function theory. The last two sections are devoted to differentiation and integration.

Some more advanced topics are also discussed in optional paragraphs, e.g. metric spaces, Weierstrass's Approximation Theorem, Riemann—Stieltjes integral, the latter in a remarkably elegant version.

Everywhere when a new property or new concept is introduced, these are illustrated by examples. This large number of examples make the book very valuable both for students and teachers. The above mentioned "new Riemann-Stieltjes integrability" will surely have a quick success among analysis lecturers. The book contains many elementary exercises — the presentation of some more advanced exercises might have still raised the value of the book.

We recommend it for students having some experience in calculus (the functions $\sin x$, e^x etc. are assumed to be familiar) and especially for teachers or future teachers in secondary schools.

V. Totik (Szeged)

L. E. Sigler, Exercises in Set Theory, Springer-Verlag, New York, Heidelberg, Berlin, 1976. pp. 133. (First published in 1966.)

This is the second edition of a group of exercises planned to help undergraduate students in studying set theory. Almost all problems are on a routine level assuming knowledge of basic set theoretic definitions and theorems included at the beginning few paragraphs of each section and, besides, some elementary notions in algebra involving the concepts of monoid, semigroup, group, ring, vector space and algebra. The selection of the topics is in accord with the structure of the volume Naive Set Theory by P. Halmos: Elementary Concepts, Cartesian Product, Relations, Functions, Families, Functions Defined on Power Sets, Applications of Functions, The Natural Numbers, Order, The Axiom of Choice and Zorn's Lemma, Well Orderings, Transfinite Recursion and Similarity, Ordinals and Cardinals. The clearly written answers are presented at the end of the book.

P. Ecsedi—Tóth (Szeged)

D. R. Smart, Fixed Point Theorems, VIII+93 pp., Cambridge University Press, Cambridge, London, New York, New Rochelle, Melbourne, Sydney, 1980 (second edition).

This is the first paperback edition of an excellent book published in 1974. The aim of the book is to treat the classical fixed point theorems and to show a number of applications in analysis.

Thus, the author begins with Banach's contraction principle; after this he proves, with a minimal use of algebraic topology, Brouwer's theorem from which he deduces other fixed point results such as Schauder's, Rothe's, Krasnoselskii's etc. The elements of fixed point theory for families of mappings and for many-valued mappings, as well as some recent results (e.g. Bowder's theorem about the existence of fixed points of non-espansive mappings of the closed unit ball in a Hilbert space) are also treated.

The applications are to the existence of solutions of differential equations; to the existence of invariant means and implicit functions; to minimax theorems etc.

The book concludes with a brief outline of numerical invariants used in fixed point theory.

The author often leaves details to the reader, by which he is able to point out the key steps in the proofs. The book contains interesting exercises and unsolved problems; the status of the latter, however, might have been mentioned in the new edition.

We recommend this introductory work to everybody who wants to get acquainted with this useful and interesting part of mathematics and with its applications.

V. Totik (Szeged)

B. R. Tennison, Sheaf Theory (London Mathematical Society Lecture Note Series, 20), VII+164 pages, Cambridge University Press, 1975.

This is a very good introduction to some questions of sheaf theory. At present sheaf theory finds its main applications in topology and in algebraic geometry, where it has been used to solve several longstanding problems. This book gives a general definition of a manifold, incorporating both the geometric and topological special cases, and then applies sheaf cohomology to such objects. "The approach to the subject taken here is rather categorical, and the course may be used (as indeed has been) as an introduction to the usefulness of categories and functors." It presupposes only the knowledge of the elements of topology and abstract algebra on the part of the reader.

A. P. Huhn (Szeged)

Jacob Wolfowitz, Selected Papers, Edited by J. Kiefer with the assistance of U. Augustine and L. Weiss, XXIII+642 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1980.

This beautiful book contains 49 papers of Professor Wolfowitz reprinted from the original journals, out of the 120 items of his bibliography published at the end of the volume. The volume appeared for the 70th birthday of the great statistician and information theorist. The editors admit that the selection of those papers that have been reprinted has been extremely difficult. "We have tried to choose the papers we regarded as most important among Wolfowitz's work in terms of their further influence, or sometimes a paper that contains what we found a striking idea of his." Following a photograph and a biographical note, the editors provide a highly clear introduction to the research works of Jacob Wolfowitz. Most of his papers (not only those reprinted) are described to some degree in this introduction.

Sándor Csörgő (Szeged)

Livres reçus par la rédaction

Analytic Functions, Proceedings of a Conference held in Kozubnik, Poland, April 19—25, 1979. Edited by L. Lawrinowicz (Lecture Notes in Mathematics, Vol. 798), 476 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1980. — DM 48,50.

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